A Nonholonomic Mechanical Structure for the Two-Dimensional Monolayer Systems

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Abstract

In this paper we study the dynamics of the 2D-motion of a particle of monolayer. A geometric approach to nonholonomic constrained mechanical systems is applied to a problem from the two dimensional geometric dynamics of the Langmuir-Blodgett monolayer. We consider a constraint submanifold of the 1-jet space $J^1(T, R^2)$, corresponding to the given constraint condition in a monolayer space and construct the corresponding constrained mechanical system on the constraint submanifold. Then the equation of motion defined on the constraint submanifold is presented.

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Introduction

Physical systems are often subjected to various types of constraints. Generally, these constraints are position, or geometric (holonomic), constraints, or velocity (nonholonomic) constraints. If a system has constraint equations that involve velocities or derivatives of system coordinates, the constraint equations are said to be nonholonomic and the mechanical system is said to be a nonholonomic system. Although almost all the work on nonholonomic systems is concerned with the case of constraints linear in components of velocities, but a geometric theory covering general nonholonomic systems has been developed by Krupkova [2], [3].

An example of such a system might be monolayer systems. In this paper we present geometric concept of the theory of nonholonomic mechanical systems developed by

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Krupkova [3] and consider the application of this theory to a problem from the two dimensional geometric dynamics of the Langmuir-Blodgett monolayer [4]. To this aim, we consider a constraint submanifold of the 1-jet space $J^1(T, R^2)$ corresponding to the given constraint conditions and construct the corresponding constrained mechanical system on the constraint submanifold. Then the equations of motion defined on the constraint submanifold are presented.

Section 3 of this paper contains a brief outline of basic geometrical concepts of nonholonomic mechanical systems that we will need. For proofs and more details see [2], [3].

Nonholonomic mechanical systems in a monolayer space

In this section we begin with a brief introduction to a monolayer space [1], [5] and then describe the nonholonomic mechanical system related to it. To obtain the 2Dmotionn equation of a particle of monolayer, we define a first order mechanical system [α] in this space and calculate the nonholonomic constrained system [α_Q] related to the mechanical system [α].

We start with the usual physical time defined by the Euclidian manifold

 $(T = [0, \infty))$ and we also consider the plane manifold R^2 having the polar coordinates (r, φ) , where r > 0 and $\varphi \in [0, 2\pi)$, and construct the 1-jet vector bundle $J^1(T, R^2) \rightarrow R \times R^2$, locally endowed with the coordinates $(t, q^1, q^2, \dot{q}^1, q^2) := (t, r, \varphi, \dot{r}, \varphi)$.

Using the special function:

$$f(z) = -\int_{-z}^{\infty} \frac{e^{-t}}{t} dt$$

we study the 2D-motion of a particle of monolayer in plane moving in such a way, that the square of its speed decreases proportionally to the reciprocal value of time passed from the beginning of the motion.

We denote by (t) the coordinate on X = T, by (t, r, φ) fibered coordinates on $Y = T \times R^2$, and by $(t, r, \varphi, r, \varphi)$ the associated coordinates on $J^1(T, R^2)$.

This particle of monolayer governed by the jet Lagrangian function $L: J^1(T, R^2) \rightarrow R$ defined by

$$L(t,r,\dot{r},\dot{\phi}) = \frac{m}{2}\dot{r}^{2} + \frac{mr^{2}}{2}\dot{\phi}^{2} \underbrace{-pr^{5}|V|e^{\frac{2|V|t}{r}} \cdot \dot{r}^{-1} + U(t,r)}_{U_{s(t,r)}},$$
(1)

where we have the following physical meanings: m is the mass of the particle; V is the *LB-monolayer compressing speed*; p is a *constant monolayer parameter* given by the physical formula:

$$p = \frac{\pi^2 q^2}{\varepsilon \varepsilon_0} \; \frac{\rho_0^2}{R_0^2};$$

 $U_s(t,r)$ is an *electro capillarity potential energy* including the monomolecular layer function:

$$U(t,r) = p \left\{ \left[-\frac{4}{3}r^5 + \frac{16}{15}(|V|t)r^4 + \frac{1}{30}(|V|t)^2r^3 + \frac{1}{45}(|V|t)^3r^2 + \frac{1}{45}(|V|t)^4r + \frac{2}{45}(|V|t)^5 \right] e^{\frac{2|V|t}{r}} - \frac{4}{45} + \frac{(|V|t)^6}{r}f(\frac{2|V|t}{r}) \right\}$$

According to the Lagrange theory on fibered manifolds [3], to every Lagrangian there exists a unique Lepagean equivalent, denoted by θ_{λ} , where λ is a Lagrangian on $J^1(T, R^2)$. The Euler-Lagrange form defined by the Lepagean equivalent depends only on the Lagrangian λ . Therefore it is referred to as the Euler-Lagrange form of the Lagrangian λ , and is denoted by E_{λ} . Now, we shall find chart expressions for the Lepagean and the Euler-Lagrange forms.

In fibered coordinates $(t, r, \varphi, r; \varphi)$ on $J^1(T, R^2)$, the module of contact 1-forms (denoted by $\Omega^1 J^1(T, R^2)$) is locally generated by the forms $\omega^1 = dr - r dt$ and $\omega^2 = d\varphi - \varphi dt$. Put $\lambda = L dt$ and denote

$$\theta_{\lambda} = L dt + \frac{\partial L}{\partial \dot{r}} \omega^{1} + \frac{\partial L}{\partial \dot{\phi}} \omega^{2}, \qquad (2)$$

accordingly

$$\theta_{\lambda} = \left(\frac{1}{2} m(\dot{r}^{2} + r^{2} \dot{\phi}^{2}) - p r^{5} |V| e^{\frac{2|V|t}{r}} \dot{r}^{-1} + U(t, r)\right) dt + \left(m \dot{r} + p r^{5} |V| e^{\frac{2|V|t}{r}} \dot{r}^{-2}\right) \omega^{1} + (m r^{2} \dot{\phi}) \omega^{2}.$$
(3)

We define a first order mechanical system $[\alpha]$ on the fibered manifold $J^1(T, R^2) \rightarrow R \times R^2$ represented by the 2-form with respect to (15)

$$\begin{aligned} \alpha &= d\theta_{\lambda} + F = \left(mr\dot{\varphi}^{2} - \frac{5p \, r^{4} \, |V|e^{\frac{2|V|t}{r}}}{\dot{r}} + \frac{2pr^{3} \, |V|^{2}t \, e^{\frac{2|V|t}{r}}}{\dot{r}} + \frac{\partial U}{\partial r} \right) dr \wedge dt \\ &+ \left(\frac{2pr^{4} \, |V|^{2}e^{\frac{2|V|t}{r}}}{\dot{r}^{2}} \right) dt \wedge \omega^{1} + \left(\frac{5pr^{4} \, |V|e^{\frac{2|V|t}{r}}}{\dot{r}^{2}} - \frac{2pr^{3} \, |V|^{2} \, te^{\frac{2|V|t}{r}}}{\dot{r}^{2}} \right) dr \wedge \omega^{1} \\ &+ \left(m - \frac{2pr^{5} \, |V|e^{\frac{2|V|t}{r}}}{\dot{r}^{3}} \right) d\dot{r} \wedge \omega^{1} + (2mr \, \dot{\phi}) dr \wedge \omega^{2} + (mr^{2}) d\dot{\phi} \wedge \omega^{2} + F. \end{aligned}$$
(4)

This mechanical system is related to the dynamical form with respect to (11)

$$E = E_1 \, dr \wedge \, dt + E_2 \, d\varphi \wedge \, dt, \tag{5}$$

where

$$E_{1} = \left(mr\dot{\phi} - \frac{10pr^{4} |V|e^{\frac{2|V|t}{r}}}{\dot{r}} + \frac{4 pt r^{3} |V|^{2} e^{\frac{2|V|t}{r}}}{\dot{r}} - \frac{2pr^{4} |V|^{2} e^{\frac{2|V|t}{r}}}{\dot{r}^{2}} + \frac{\partial U}{\partial r} \right) - \left(m + \frac{2pr^{5}|V|e^{\frac{2|V|t}{r}}}{\dot{r}^{2}} \right) r,$$

and

$$E_2 = -(2mrr + mr^2\varphi)$$

We consider the nonholonomic constraint Q given by the equation

$$f(t,r,\varphi,r;\varphi) \equiv [(r)^{2} + (\varphi)^{2}] - \frac{1}{t} = 0,$$
(6)

which means that the particle's speed decreases proportionally to $\frac{1}{\sqrt{t}}$. In a neighborhood of the submanifold Q

$$rank\left(\frac{\partial f}{\partial \dot{r}}, \frac{\partial f}{\partial \varphi}\right) = rank\left(2\dot{r}, 2\dot{\varphi}\right) = 1.$$
(7)

Let $U \subset J^1(T, R^2)$ be the set of all points, where $\dot{\phi} > 0$, and consider on U the adapted coordinates $(t, r, \phi, r; \bar{f})$, where $\bar{f} = \dot{\phi} - g$, $g = \sqrt{\frac{1}{t} - (\dot{r})^2}$ is the equation of the constraint (6) in normal form.

The constrained system $[\alpha_Q]$ related to the mechanical system $[\alpha]$ and the constraint Q is the equivalence class of the 2-form (with respect to (18))

$$\alpha_{Q} = A'_{1} \omega^{1} \wedge dt + B'_{1,1} \omega^{1} \wedge d\dot{r} + \bar{F} + \phi_{(2)}$$
(8)

on Q, where \overline{F} is any 2-contact 2-form and $\phi_{(2)}$ is any constraint 2-form defined on Q. Calculating $\overline{L} = L \circ \iota$ and calculating $A'_1, B'_{1,1}$ by relationships (19), (20)

$$\overline{\mathcal{L}}(t,r,\varphi,\dot{r},\dot{\varphi}) = L\left(t,r,\varphi,\dot{r},\sqrt{\frac{1}{t}-(\dot{r})^2}\right).$$

Then

$$\bar{\mathbf{L}} = \frac{m}{2}\dot{r}^{2} + \frac{mr^{2}}{2}\left(\frac{1}{t} - \dot{r}^{2}\right) \underbrace{-pr^{5}|V|e^{\frac{2|V|t}{r}}}_{U_{s(t,r)}} \dot{r}^{-1} + U(t,r),$$

and

$$A_{1}^{'} = mrg^{2} + 2mr\dot{r}^{2} - \frac{m\dot{r}r^{2}}{2t^{2}}g^{2} - \frac{10pr^{4}t|V|^{2}}{\dot{r}}e^{\frac{2|V|t}{r}} + \frac{4pr^{3}t|V|^{2}}{\dot{r}}e^{\frac{2|V|t}{r}} - \frac{2pr^{4}|V|^{2}e^{\frac{2|V|t}{r}}}{\dot{r}^{2}} + \frac{\partial U}{\partial r'}$$

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$$B'_{1,1} = m(r^2 - 1) - \frac{mr^2}{tg^2} + \frac{2pr^5t|V|e^{\frac{2|V|t}{r}}}{r^3}$$

Reduced equation of motion of the constrained system is as follows

$$[A'_{1} + B'_{1,1} r] \circ J^{2} \bar{\gamma} = 0, \qquad (9)$$

where $\bar{\gamma} = (t, r(t), \varphi(t))$ is a section satisfying the constraint equation $f \circ J^1 \gamma = 0$.

Lagrangian systems on fibered manifolds

Throughout this section we consider a fibered manifold $\pi: Y \to X$ with a one dimensional base space X and (m + 1) -dimensional total space Y. We use jet prolongations $\pi_1: J^1(X,Y) \to X$ and $\pi_2: J^2(X,Y) \to X$ and jet projections $\pi_{1,0}: J^1(X,Y) \to Y$ and $\pi_{2,1}: J^2(X,Y) \to J^1(X,Y)$. Configuration space at a fixed time is represented by a fiber of the fibered manifold π and a corresponding phase space is then a fiber of the fibered manifold π_1 . Local fibered coordinates on Y are denoted by (t, q^{σ}) , where $1 \leq \sigma \leq m$. The associated coordinates on $J^1(X,Y)$ and $J^2(X,Y)$ are denoted by $(t, q^{\sigma}, q^{\sigma})$ and $(t, q^{\sigma}, \dot{q}^{\sigma}, q^{\sigma})$, respectively. In calculations we use either a canonical basis of one forms on $J^1(X,Y)$, $(dt, dq^{\sigma}, d\dot{q}^{\sigma})$, or a basis adapted to the contact structure

$$(dt, \omega^{\sigma}, d\dot{q}^{\sigma}),$$

where

$$\omega^{\sigma} = dq^{\sigma} - \dot{q}^{\sigma} dt, \qquad 1 \le \sigma \le m.$$

Whenever possible, the summation convention is used. If $f(t, q^{\sigma}, \dot{q}^{\sigma})$ is a function defined on an open set of $J^{1}(X, Y)$ we write

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial q^{\sigma}} \dot{q}^{\sigma} + \frac{\partial f}{\partial \dot{q}^{\sigma}} q^{\sigma}, \qquad \frac{\bar{d}f}{\bar{d}t} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial q^{\sigma}} \dot{q}^{\sigma}.$$

A differential form ρ is called *contact* if $J^1\gamma^*\rho = 0$ for every section γ of π . A contact 2 -form ρ is called *1-contact* if for every vertical vector field ξ , $i_{\xi}\rho_1$ is a horizontal; ρ is 2 -contact if $i_{\xi}\rho_1$ is 1-contact. The operator assigning to ρ it's 1-contact part is denoted by p_1 .

If λ is a Lagrangian on $J^1(X, Y)$, we denote by θ_{λ} its *Lepage equivalent* or *Cartan* form and E_{λ} its Euler-Lagrange form, respectively. Recall that $E_{\lambda} = p_1 d\theta_{\lambda}$. In fibered coordinates where $\lambda = L(t, q^{\sigma}, \dot{q}^{\sigma}) dt$, we have

$$\theta_{\lambda} = L \, dt \, + \frac{\partial L}{\partial \dot{q}^{\sigma}} \, \omega^{\sigma}, \tag{10}$$

and

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$$E_{\lambda} = E_{\sigma} \left(L \right) \omega^{\sigma} \wedge dt, \tag{11}$$

where the components $E_{\sigma}(L) = \frac{\partial L}{\partial q^{\sigma}} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}^{\sigma}}$, are the Euler-Lagrange expressions.

Since the functions E_{σ} are affine in the second derivatives we write

$$E_{\sigma} = A_{\sigma} + B_{\sigma\nu} q^{\nu}$$

where

$$A_{\sigma} = \frac{\partial L}{\partial q^{\sigma}} - \frac{\partial^2 L}{\partial t \,\partial \dot{q}^{\sigma}} - \frac{\partial^2 L}{\partial q^{\nu} \partial \dot{q}^{\sigma}} \,\dot{q}^{\nu}, \qquad \qquad B_{\sigma\nu} = -\frac{\partial^2 L}{\partial \dot{q}^{\sigma} \,\partial \dot{q}^{\nu}}. \tag{12}$$

A section γ of π is called a *path* of the Euler-Lagrange form E_{λ} if

$$E_{\lambda} \circ J^2 \gamma = 0. \tag{13}$$

In fibered coordinates this equation represents a system of m second-order ordinary differential equations

$$A_{\sigma}\left(t,\gamma^{\nu},\frac{d\gamma^{\nu}}{dt}\right) + B_{\sigma}\rho\left(t,\gamma^{\nu},\frac{d\gamma^{\nu}}{dt}\right)\frac{d^{2}\gamma^{\rho}}{dt^{2}} = 0, \qquad (14)$$

for components $\gamma^{\nu}(t)$ of a section γ , where $1 \le \nu \le m$. These equations are called *Euler-Lagrange equations* or *motion equations* and their solutions are called *paths*.

Euler-Lagrange equations (14) can be written in the form

$$J^1 \gamma^* i_{\xi} \alpha = 0,$$

where $\alpha = d\theta_{\lambda} + F$ is any 2-form defined on an open subset $W \subset J^{1}(X, Y)$, such that $p_{1}\alpha = E_{\lambda}$, and F is a 2-contact 2-form. In fibered coordinates we have $F = F_{\sigma\nu}\omega^{\sigma} \wedge \omega^{\nu}$, where $F_{\sigma\nu}(t, q^{\rho}, q^{\rho})$ are arbitrary functions. Recall from [3] that the family of all such (local) 2-forms:

$$\alpha = d\theta_{\lambda} + F = A_{\sigma}\omega^{\sigma} \wedge dt + B_{\sigma\nu}\omega^{\sigma} \wedge dq^{\nu} + F$$
(15)

is called a *first order Lagrangian system*, and is denoted by $[\alpha]$.

A non-holonomic constrained mechanical system is defined on the (2m+1-k)dimensional constraint submanifold $Q \subset J^1(X, Y)$ fibered over Y and given by k equations

$$f^{i}(t,q^{1},...,q^{m},q^{1},...,\dot{q}^{m}) = 0, \qquad 1 \le i \le k,$$

where

$$rank \left(\frac{\partial f^{i}}{\partial \dot{q}^{\sigma}}\right) = k, \tag{16}$$

or following [6], equivalently in an explicit form

$$\dot{q}^{m-k+i} = g^{i}(t, q^{\sigma}, \dot{q}^{1}, \dot{q}^{2}, \dots, \dot{q}^{m-k}), \qquad 1 \le i \le k.$$
(17)

By a nonholonomic constrained system arising from the Lagrangian system $[\alpha]$ and constraint forms on the constraint submanifold Q, we mean the equivalence class $[\alpha_Q]$ on Q, where

$$\alpha_Q = \iota^* d\theta_\lambda + \bar{F} + \phi_{(2)}$$

where \overline{F} is any 2-contact $\pi_{1,0}$ horizontal 2-form and ϕ_2 is any constraint 2-form defined on Q, and ι is the canonical embedding of Q into $J^1(X, Y)$. The local form of $[\alpha_Q]$ is

$$\alpha_{Q} = \sum_{l=1}^{m-k} A_{l}^{'} \omega^{l} \wedge dt + \sum_{l,s=1}^{m-k} B_{ls}^{'} \omega^{l} \wedge d\dot{q}^{s} + \bar{F} + \phi_{(2)}, \qquad (18)$$

where the components $A_{l}^{'}$ and $B_{l,s}^{'}$ are given by

$$\begin{split} A_{l}^{'} &= \frac{\partial \bar{L}}{\partial q^{l}} + \frac{\partial \bar{L}}{\partial q^{m-k+i}} \frac{\partial g^{i}}{\partial \dot{q}^{l}} - \frac{\bar{d}c}{dt} \frac{\partial \bar{L}}{\partial \dot{q}^{l}} + \left(\frac{\partial L}{\partial \dot{q}^{m-k+j}}\right)_{l} \left[\frac{\bar{d}c}{dt} \left(\frac{\partial g^{j}}{\partial \dot{q}^{l}}\right) - \frac{\partial g^{j}}{\partial q^{l}} - \frac{\partial g^{j}}{\partial q^{m-k+i}} \frac{\partial g^{i}}{\partial \dot{q}^{l}}\right], \end{split}$$

$$B_{l,s}^{'} = -\frac{\partial^2 \bar{L}}{\partial \dot{q}^l \partial \dot{q}^s} + \left(\frac{\partial L}{\partial \dot{q}^{m-k+i}}\right)_l \frac{\partial^2 g^i}{\partial \dot{q}^l \partial \dot{q}^s}$$

 $\overline{L} = L \circ \iota$, and

$$\frac{\bar{d}c}{dt} = \frac{\partial}{\partial t} + \dot{q}^s \frac{\partial}{\partial q^s} + g^i \frac{\partial}{\partial q^{m-k+i}}$$

The equations of the motion of the constrained system [α_Q] in fibered coordinates take the form

$$\left(A_{l}' + \sum_{s=1}^{m-k} B_{l,s}' \ddot{q}^{s}\right) \circ J^{2} \bar{\gamma} = 0.$$
(19)

for components $\gamma^1(t), \gamma^2(t), \dots, \gamma^{m-k}(t)$ of a Q – admissible section $\bar{\gamma}$ dependent on time t and parameters $q^{m-k+1}, q^{m-k+2}, \dots, q^m$, which have to be determined as functions $\gamma^{m-k+1}(t), \gamma^{m-k+2}(t), \dots, \gamma^m(t)$ from the equations (17) of the constraint

$$\frac{dq^{m-k+i}}{dt} = g^i \left(t, q^{\sigma}, \frac{dq^1}{dt}, \frac{dq^2}{dt}, \dots, \frac{dq^{m-k}}{dt} \right), \quad 1 \le i \le k.$$

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