On the Irreducible Characters of Camina Triples

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Extended Abstract

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Introduction

Let G be a finite group. We say that the group G has the Camina triple property if there exist normal subgroups M and N of G with $N \leq M$ such that for every $g \in G \setminus M$ the conjugacy class of g in G is equal to gN. In this case, (G,M,N) is called the Camina triple. In the special case when $M = N = G'$, the group G is a Camina group. This shows that the Camina triple property is a generalization of the Camina property in finite groups. In recent years, the groups with the Camina triple property have been studied in some special cases. For instance, the Camina triples $(G,Z(G),G')$ and $(G,Z(G)G',G')$ have been studied by Lewis. In fact, in both cases Lewis has given a characterization of G in terms of its irreducible characters and moreover, he determined the irreducible characters of G in terms of irreducible characters of G/N and M. In this paper, we determine the irreducible characters of a Camina triple (G,M,N) in terms of the irreducible characters of G/N and M.

Material and methods

Let G be a finite group. It is known that there is a one-to-one correspondence between the set of irreducible characters of G and the set of irreducible characters of $Z(\mathbb{C}G)$, the center of the group algebra $\mathbb{C}G$. In fact $\mathrm{Irr}(Z(\mathbb{C}G)) = \{\omega_\chi \mid \chi \in \mathrm{Irr}(G)\}$ such that $\omega_\chi(\overline{\mathrm{Cl}_G(g)}) = \dfrac{|\mathrm{Cl}_G(g)|\chi(g)}{\chi(1)}$ where $\overline{\mathrm{Cl}_G(g)} = \sum_{h \in \mathrm{Cl}_G(g)} h$. More precisely, if $T = (\chi_i(g_j))_{0 \leq i,j \leq h}$ is the character table of G and P is the character table of $Z(\mathbb{C}G)$, then we have

$$T = \begin{pmatrix} \chi_1(1) & & 0 \\ & \ddots & \\ 0 & & \chi_h(1) \end{pmatrix} .P. \begin{pmatrix} \dfrac{1}{|\mathrm{Cl}_G(g_0)|} & & 0 \\ & \ddots & \\ 0 & & \dfrac{1}{|\mathrm{Cl}_G(g_h)|} \end{pmatrix}.$$

Now let (G,M,N) be a Camina triple. To determine the character table of G we first determine P and then from the above equality we find T. Here we mention that if $\chi \in \mathrm{Irr}(G/N)$ then there exists $\bar{\chi} \in \mathrm{Irr}(G)$ such that $\bar{\chi}(g) = \chi(gN)$. So we must find all irreducible characters χ of $Z(\mathbb{C}G)$ such that $N \not\subseteq \ker \chi$. To do this, for every $\psi \in \mathrm{Irr}(M)$ such that $N \not\subseteq \ker \psi$, we find an irreducible character $\bar{W}_\psi \in \mathrm{Irr}(Z(\mathbb{C}G))$. Moreover, we show that if ω_φ is an irreducible character of $Z(\mathbb{C}G)$ such that $N \not\subseteq \ker\varphi$, then $\omega_\varphi = \bar{W}_\psi$ for some $\psi \in \mathrm{Irr}(G)$. Hence we are able to determine the character table P of $Z(\mathbb{C}G)$. So the character table T of G can be found.

Results and discussion

Let (G,M,N) be a Camina triple. Suppose that $\psi \in \mathrm{Irr}(M)$ such that $N \not\subseteq \ker\psi$. We define the map $\bar{W}_\psi : Z(\mathbb{C}G) \to \mathbb{C}$ such that

$$\begin{cases} \bar{W}_\psi(\overline{\mathrm{Cl}_G(g)}) = W_\psi(\overline{\mathrm{Cl}_G(g)}), & g \in M \\ \bar{W}_\psi(\overline{\mathrm{Cl}_G(g)}) = 0, & g \in G \setminus M \end{cases}$$

where $W_\psi(\mathrm{Cl}_G(g)) = \sum_{\mathrm{Cl}_M(m) \subseteq \mathrm{Cl}_G(g)} \omega_\psi(\mathrm{Cl}_M(m))$. We first show that $\bar{W}_\psi \in \mathrm{Irr}(Z(\mathbb{C}G))$ and if $\{\psi_1,\ldots,\psi_r\}$ is the set of all irreducible characters of M such that $N \not\subseteq \ker\psi_i$ and for every $1 \le i,j \le r$ with $i \ne j$, ψ_i and ψ_j are not $G-$conjugate, then $\{\bar{W}_{\psi_1},\ldots,\bar{W}_{\psi_r}\}$ is a subset of irreducible characters of $Z(\mathbb{C}G)$. Moreover, we show that if ω_φ is an irreducible character of $Z(\mathbb{C}G)$ such that $N \not\subseteq \ker\varphi$, then $\omega_\varphi = \bar{W}_{\psi_i}$ for some $1 \le i \le r$.

The main result of this paper is as follows.

Let (G,M,N) be a Camina triple. Let $\mathrm{Irr}(G/N) = \{\chi_1,\ldots,\chi_m\}$ and $\{\psi_1,\ldots,\psi_r\}$ be the set of all irreducible characters of M such that $N \not\subseteq \ker\psi_i$ and for every $1 \le i,j \le r$ with $i \ne j$, ψ_i and ψ_j are not $G-$conjugate. Then $\mathrm{Irr}(G) = \{\bar{\chi}_1,\ldots,\bar{\chi}_m,\bar{\psi}_1,\ldots,\bar{\psi}_r\}$ such that $\bar{\chi}_i(g) = \chi_i(gN)$ and

$$\begin{cases} \bar{\psi}_i(g) = 0, & g \in G \setminus M \\ \bar{\psi}_i(g) = \dfrac{|G|\psi_i(1)}{|\mathrm{Cl}_G(g)|\sqrt{|M||I_G(\psi_i)|}} W_{\psi_i}(\overline{\mathrm{Cl}_G(g)}), & g \in M \end{cases}$$

where $W_{\psi_i}(\overline{\mathrm{Cl}_G(g)}) = \sum_{\mathrm{Cl}_M(m) \subseteq \mathrm{Cl}_G(g)} \omega_{\psi_i}(\overline{\mathrm{Cl}_M(m)})$.

Conclusion

Let (G,M,N) be a Camina triple. In this paper we have determined the character table of G in terms of the irreducible characters of G/N and M. To do this we first obtained the character table of $Z(\mathbb{C}G)$. Then, since there is a one-to-one correspondence between the set of irreducible characters of G and the set of irreducible characters of $Z(\mathbb{C}G)$ the character table of G was found.

Keywords: Finite group; Character; Camina triple.

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