Existence of Two Solutions for Nonlinear Difference Equations Involving p(k)-Laplacian Operator and Boundary Value Conditions

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Extended Abstract

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Introduction

There seems to be increasing interest in the existence of solutions to boundary value problems for finite difference equations with p(k)-Laplacian operator, because of their applications in many fields. Results on this topic are usually achieved by using various fixed point theorems in cone. This kind of problems play a fundamental role in different fields of research, such as mechanical engineering, control systems, economics, computer science, physics, artificial or biological neural networks, cybernetics, ecology and many others. Important tools in the study of nonlinear difference equations are fixed point methods and upper and lower solution techniques. It is well known that critical point theory is an important tool to deal with the problems for differential equations. More, recently, the existence and multiplicity of solutions for nonlinear discrete boundary value problems have been investigated by adopting variational methods.

The aim of this paper is to establish the existence of positive solutions for the following discrete boundary-value problem

$$(P_\lambda^f)\begin{cases} -\Delta(w(k-1)\phi_{p(k-1)}(\Delta u(k-1))) + q(k)\phi_{p(k)}(u(k)) = \lambda f(k,u(k)), \\ u(0) = u(T+1) = 0, \end{cases} \tag{1}$$

for every $k \in [1,T]$, where $k \in [1,T]$ is a fixed positive integer, λ is a positive real parameter, $[1,T]$ is the discrete interval $\{1,\dots,T\}$, $p:[0,T+1]\to[2,\infty)$, $w:[0,T]\to[1,\infty)$ and $q:[0,T+1]\to[1,\infty)$, are given functions and $\Delta u(k) = u(k+1)-u(k)$ is the forward difference operator, $\phi_{p(\cdot)}(s) = |s|^{p(\cdot)-2}s$ is the one-dimensional discrete $p(\cdot)$-Laplacian operator and $f:[1,T]\times\mathbb{R}\to\mathbb{R}$ is the continuous function. Throughout the text we will use the following notations

$$p^+ := \max_{k\in[0,T+1]} p(k), \qquad p^- := \min_{k\in[0,T+1]} p(k), \qquad \overline{w} = \sum_{k=1}^{T+1} w(k-1), \qquad \overline{q} = \sum_{k=1}^{T+1} q(k),$$

Based on a local minimum theorem, we ensure an exact interval of the parameter λ, in which problem (1) admits at least two solutions. As an example, here, we point out the following special case of our main result.

Theorem 1 Let $T \geq 2$ be a fixed positive integer and let $f:\mathbb{R}\to\mathbb{R}$ be continues and non-nagative function and

$$\limsup_{\xi\to 0}\frac{\int_0^{\xi} f(s)ds}{|\xi|^{T+3}} < \frac{5^{-\frac{T+3}{2}}T^{-\frac{T+1}{2}}}{0.5(T+3)(T+1)}\liminf_{\xi\to\infty}\frac{1}{|\xi|^{T+3}}\int_0^{\xi} f(t)dt.$$

Then for any

$$\lambda\in\Lambda^{*} = \left]\frac{5^{\frac{T+3}{2}}T^{\frac{T+1}{2}}(T+1)}{2\liminf_{\xi\to\infty}\frac{1}{|\xi|^{T+3}}\int_0^{\xi} f(t)dt}, \frac{1}{(T+3)\limsup_{\xi\to 0}\frac{\int_0^{\xi} f(s)ds}{|\xi|^{T+3}}}\right[,$$

the problem

$$\begin{cases} -\Delta(|\Delta u(k-1)|^{k-1}\Delta u(k-1)) = \lambda f(u(k)) - (u(k))^{k+1}, \quad k\in 1,T], \\ u(0) = u(T+1) = 0, \end{cases} \tag{2}$$

admits at least two solutions $u_{\lambda,1}, u_{\lambda,2}\in W$.

In this paper, we recall some basic definitions and the main tools which we use to show our results. We also provide a few inequalities which play very important role in our investigations and we provide our main result that contains the existence theorem, some examples are also provided.

Material and methods

Before providing the mentioned result we recall that a continuously differentiable functional I defined on a real Banach space X satisfies the Palais-Smale condition, if every sequence $\{u_n\}$ such that $\{I(u_n)\}$ is bounded and $\{I^{'}(u_n)\}$ covergence to zero in X^{*}, has a convergent subsequence.

As a special case of mountain pass theorem, we present the following version:

Theorem 2 Let X be a real finite dimensional Banach space and $I: X\to\mathbb{R}$ a continuously Gateaux differentiable functional. Assume that I satisfies Palais-Smale condition and it is unbounded from below. Further assume that I admits a local minimum u_1. Then, I admits a distinct second critical point.

In order to give the variational formulation of problem (1) we introduce the T-dimensional Banach space $W := \{u: [0, T+1]\to\mathbb{R} : u(0) = u(T+1) = 0\}$, which is a Hilbert space under the norm

$$\|u\| = \left\{\sum_{k=1}^{T+1} w(k-1)|\Delta u(k-1)|^{p^-} + q(k)|u(k)|^{p^-}\right\}^{1/p^-}.$$

Since W is finite-dimensional, we can also define the following equivalent norm on W

$$\|u\|_{+} = \left\{\sum_{k=1}^{T+1} w(k-1)|\Delta u(k-1)|^{p^+} + q(k)|u(k)|^{p^+}\right\}^{1/p^+}.$$

Now, let $\psi: W\to\mathbb{R}$ be given by the formula

$$\psi(u) := \sum_{k=1}^{T+1}[w(k-1)|\Delta u(k-1)|^{p(k-1)} + q(k)|u(k)|^{p(k)}].$$

To study problem (1) we consider the functional $I_\lambda: W \to \mathbb{R}$ defined by

$$I_\lambda(u) := \sum_{k=1}^{T+1}\left(\frac{w(k-1)}{p(k-1)}|\Delta u(k-1)|^{p(k-1)} + \frac{q(k)}{p(k)}|u(k)|^{p(k)}\right) - \lambda\sum_{k=1}^{T}F(k,u(k)).$$

It is clear that I_λ turns out to be of class C^1 on W such that for all $u, v \in W$.

$$I'_\lambda(u)(v) := \sum_{k=1}^{T+1}\Big(w(k-1)\phi_{p(k-1)}(\Delta u(k-1))\Delta v(k-1) + q(k)\phi_{p(k)}(u(k))v(k)\Big) - \sum_{k=1}^{T}\lambda f(k,u(k))v(k)$$

The critical points of I_λ coincide exactly with the solutions of boundary value problem (1). To describe the variational framework of problem (1,1) put Φ, Ψ as follows

$$\Psi(u) := \sum_{k=1}^{T}F(k,u(k)), \quad \Phi(u) := \sum_{k=1}^{T+1}\left(\frac{w(k-1)}{p(k-1)}|\Delta u(k-1)|^{p(k-1)} + \frac{q(k)}{p(k)}|u(k)|^{p(k)}\right),$$

for every $u \in W$. Then $I_\lambda = \Phi - \lambda\Psi$. Put

$$L_\infty := \min_{k\in[1,T]}\left(\liminf_{\xi\to\infty}\frac{F(k,\xi)}{|\xi|^{p^+}}\right), \qquad \lambda^* := \frac{(T)^{\frac{p^+-2}{2}}(4\overline{w}+\overline{q})^{\frac{p^+}{2}}}{L_\infty p^-\left(\max\{\overline{w},\overline{q}\}\right)^{\frac{p^+-2}{2}}}.$$

Lemma 1 Let $f:[1,T]\times\mathbb{R}\to\mathbb{R}$ be a continuous function, such that $f(k,x)\ge 0$ for all $x \le 0$ and for all $k\in[1,T]$. If $L_\infty > 0$ then the functional I_λ satisfies the Palais-Smale condition and it is unbounded from below for all $\lambda \in]\lambda^*, +\infty[$.

Now we give a lemma and the following notation. Put

$$L^0 := \max_{k\in[1,T]}\limsup_{\xi\to 0}\frac{F(k,\xi)}{|\xi|^{p^+}}$$

Lemma 2 If $0 \le L^0 < \infty$ then 0 is a local minimum for I_λ for each $\lambda \in]0, \frac{1}{p^+L^0}[$.

Results and discussion

To show that problem (1) has multiple solutions, precisely it has at least two positive solutions, we apply Theorem above. Now, we provide the main result of this section.

Theorem 3 Let $f:[1,T]\times\mathbb{R}\to\mathbb{R}$ be a non-negative continuous function, such that $f(k,x) = f(k,0)$ for all $x \le 0$ and for all $k \in [1,T]$. If $L_\infty > 0$ and

$$\lambda^* < \frac{1}{p^+L^0}, \tag{3}$$

then problem (1) has at least two positive solutions $u_{\lambda,1}, u_{\lambda,2} \in W$ for each $\lambda \in \Lambda :=]\lambda^*, \frac{1}{p^+L^0}[$.

Conclusion

Next example illustrates Theorem above as conclusion.

Example 1 Let $T = 10$ and for any $x \in \mathbb{R}$ and $k \in [1,T]$

$$f(k,x) = \begin{cases} 4x^3\cosh(6\tan^{-1}x) + \dfrac{6x^4}{x^2+1}\sinh(6\tan^{-1}x) & x \ge 0 \\ 0 & x \le 0 \end{cases}$$

and $w(k) = k^3$, $q(k) = \binom{T}{k-1}$, $p(k) = \sqrt{k+5}$, hence $F(k,x) = x^4\cosh(6\tan^{-1}x)$

$$\bar{q} = \sum_{k=1}^{T+1} \binom{T}{k-1} = \sum_{k=0}^{T} \binom{T}{k} = 2^T = 1024, \quad \bar{w} = \sum_{k=1}^{T+1} (k-1)^3 = \left(\frac{T(T+1)}{2}\right)^2 = 3025$$

$$p^+ = 4, \quad p^- = \sqrt{5}\,, \quad L^0 = \lim_{x\to 0} \cosh(6\tan^{-1} x) = 1$$

$$L_\infty = \lim_{x\to\infty} \cosh(6\tan^{-1} x) = \cosh(3\pi)\,, \lambda^* = \frac{10(4\times 3025 + 1024)^2}{(3025)^2\sqrt{5}\cosh(3\pi)} = 0.013586$$

$$\frac{1}{p^+ L^0} = 0.24$$

Hence (3) holds. Then by Theorem 3, the problem

$$\begin{cases} -\Delta\left((k-1)^3 \phi_{\sqrt{k+4}}(\Delta u(k-1))\right) + \binom{T}{k-1} \phi_{\sqrt{k+5}}(u(k)) = \lambda f(k, u(k)), & k \in [1,10] \\ u(0) = u(11) \end{cases}$$

has at least two non-zero solutions $u_{\lambda,1}, u_{\lambda,2} \in W$ for any $\lambda \in \Lambda := [0.013586\ , 0.24]$.

Example 2 Let $T \geq 2$ and $F(k,x) = x^{T+3}e^x$ for any $k \in [1,T]$, since

$$f(k,x) = \begin{cases} x^{T+2}e^x(x+T+3), & x \geq 0, \\ 0 & x \leq 0. \end{cases}$$

Also the inequality in Theorem 1 holds, since

$$\liminf_{\xi\to\infty} \frac{1}{|\xi|^{T+3}} \int_0^\xi f(t)dt = +\infty \text{ and } \limsup_{\xi\to 0} \frac{\int_0^\xi f(s)ds}{|\xi|^{T+3}} = 1.$$

Then by Theorem 1, for any $\lambda \in \left]0, \frac{1}{T+3}\right[$, the problem (2) admits at least two solutions $u_{\lambda,1}, u_{\lambda,2} \in W$.

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