Adjunctions Between Hom and Tensor as Endofunctors of (bi-)Module Category of Comodule Algebras Over a Quasi-Hopf Algebra

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Extended Abstract

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Introduction

Over a commutative ring k, it is well known from the classical module theory that the tensor-endofunctor of $\mathbb{M}_k$ is left adjoint to the Hom-endofunctor. The unit and counit of this adjunction is obtained trivially.

For a k-bialgebra (H, , , ,) the category of (H,H)-bimodules is a monoidal category: the tensor product $M\otimes_k N$ of two arbitrary (H,H)-bimodules M and N is again an (H,H)-bimodule in which the bimodule structure of $M\otimes_k N$ is defined diagonally using the comultiplication. The associativity constraint of this category is formally trivial as in the category $\mathbb{M}_k$ and it is followed from the coassociativity of . An antipode is an algebra anti-homomorphism S;H→H which is the inverse of id_H with respect to the convolution product in $End_k(H)$. A Hopf algebra is a bialgebra together with an antipode.

As generalizations of the concepts bialgebra and Hopf algebra, V. G. Drinfeld introduced the concepts quasi-bialgebra and quasi-Hopf algebra respectively. A quasi-bialgebra over a commutative ring k is an associative algebra H with unit together with a comultiplication : $H\to H\otimes_k H$ and a counit : H→k satisfying all axioms of bialgebras except the coassociativity of . However, the non-coassociativity of has been controlled by a normalized 3-cocycle ∈ $H\otimes_k H \otimes_k H$ in such a way that the category ${}_H\mathbb{M}_H$ of (H,H)-bimodules is monoidal. In this case, the associativity constraint of the category is not the trivial one and it depends on the element ∈ $H\otimes_k H \otimes_k H$. However, we can yet consider tensor functors $V\otimes_k -$ and $-\otimes_k V$ as endofunctors of ${}_H\mathbb{M}_H$. A quasi-antipode has been defined as a generalization of antipode. A quasi-Hopf algebra is a quasi-bialgebra together with a quasi-antipode (S,α,β).

Let (H, , , , ,S,α,β) be a quasi-Hopf algebra with a bijective quasi-antipode S. Then it has been shown that the tensor endofunctors $V\otimes_k -$ and $-\otimes_k V$ of ${}_H\mathbb{M}_H$ have right adjoints which are described in terms of Hom-functors. This means that ${}_H\mathbb{M}_H$ is a biclosed monoidal category.

Over a Hopf algebra H, the category ${}^H\mathbb{M}$ of left H-comodules is monoidal and algebras and coalgebras can be defined inside this category. In this way, a left H-comdule algebra is defined as an algebra in the monoidal category ${}^H\mathbb{M}$ of left H-comodules. However, if H is a quasi-bialgebra or even a quasi-Hopf algebra, because of non-coassociativity of , we can not define an H-comodule algebra in this categorical language. To solve this problem, F. Hausser and F. Nill defined an H-comodule algebra in a formal way as a generalization of the quasi-bialgebra H and they considered some categories related to an H-comodule algebra such as the category of two-sided Hopf modules.

In this article, the bimodule category ${}_A\mathbb{M}_A$ of a comodule algebra A over a quasi-Hopf algebra H is considered which is not necessarily monoidal. However, we define varieties of Tensor and Hom-endofunctors of this category and state Hom-tensor adjunctions between suitable pairs of these functors. In each case, we compute the unit and counit of adjunction explicitly.

Material and methods

First we consider the category ${}_B\mathbb{M}$ of left B-modules, where B is a left comodule algebra over a quasi-Hopf algebra H and we note that the left action of ${}_H\mathbb{M}$ on ${}_B\mathbb{M}$ yields some varieties of Tensor and Hom-endofunctors of ${}_B\mathbb{M}$ and we observe that every Tensor functor defined in this way has a right adjoint which is described as a Hom-functor. Next we extend this idea for the bimodule category ${}_B\mathbb{M}_B$.

Results and discussion

First we note that although bimodule category ${}_A\mathbb{M}_A$ of a comodule algebra A over a quasi-Hopf algebra H is not monoidal, the coaction of H on A yields an action of the bimodule category ${}_H\mathbb{M}_H$ (which is monoidal) on this bimodule category. This action, in turn, allows us to define Tensor and Hom-functors as endofunctors of the bimodule category ${}_A\mathbb{M}_A$.

In any case we obtain Tensor and Hom-endofunctors with the bimodule structure defined diagonally using the coation of H on A and the quasi-antipode (S,α, β) of H. After that we state Hom-Tensor adjunction between corresponding pairs of Hom and Tensor endofunctors. The units and counits of adjunctions are not trivial as in the Hopf algebra case and they strongly depend on the invariants of the comodule algebra A and the quasi-antipode (S,α, β).

Conclusion

The following conclusions were drawn from this research.

- Let H be a quasi-Hopf algebra with the quasi-antipod (S,α,β), (B,$\boldsymbol{\lambda}$,ϕ_λ) a left H-comodule algebra and V be an (H,H)-bimodule. Then the pair

$$V\otimes_k^{b'}-:{}_B\mathbb{M}_B\to{}_B\mathbb{M}_B,\qquad {}^{t'}Hom_k^{s'}(V,-):{}_B\mathbb{M}_B\to{}_B\mathbb{M}_B$$

is an adjoint pair of endofuntors with unit and counit given by

$$\eta'_M:M\to{}^{t'}\,Hom_k^{s'}(V,V\otimes_k M),\;\; m\mapsto[\upsilon\mapsto p_\lambda.(\upsilon\otimes m).q_\lambda]$$

$$\varepsilon'_M:V\otimes{}^{t'}Hom_k^{s'}(V,M)\to M,\quad \upsilon\otimes f\mapsto\sum q_\lambda^2.[f(S^{-1}(p_\lambda^1).\upsilon.S(p_\lambda^1))].p_\lambda^2$$

where $p_\lambda=\sum p_\lambda^1\otimes p_\lambda^2$ and $q_\lambda=\sum q_\lambda^1\otimes q_\lambda^2$ are elements in H⊗B whose components are given in terms of quasi-antipode (S,α, β) and components of ϕ_λ.

- Let H be a quasi-Hopf algebra with quasi-antipod (S,α,β), (A,ρ,ϕ_ρ) a right H-comodule algebra and V be an (H,H)-bimodule. Then the pair

$$-\otimes_k^{b'}V\;:{}_A\mathbb{M}_A\to{}_A\mathbb{M}_A,\qquad {}^{s'}Hom_k^{t'}(V\,,-):{}_A\mathbb{M}_A\to{}_A\mathbb{M}_A$$

is an adjoint pair of endofuntors with unit and counit given by

$$\eta'_M:M\to{}^{s'}Hom_k^{t'}(V,M\otimes_k V),\;\; m\mapsto[\upsilon\mapsto p_\rho.(\upsilon\otimes m).q_\rho]$$

$\varepsilon_M^{'} : {}^{s^{'}}Hom_k^{t^{'}}(V,M)\otimes V \to M, \quad f\otimes v \mapsto \sum q_\rho^1.[f(S\ (q_\rho^2).\upsilon.S^{-1}(p_\rho^2))].p_\lambda^1$

where $p_\rho = \sum p_\rho^1 \otimes p_\rho^2$ and $q_\rho = \sum q_\rho^1 \otimes q_\rho^2$ are elements in A⊗H whose components are given in terms of quasi-antipode (S,α, β) and components of ϕ_ρ.

Keywords: (quasi-) Hopf algebra; Comodule algebra; Monoidal category; Action of monoidal category.

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