Numerical Stability in Complex Summation

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Extended Abstract

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Introduction

Loss of significant digits or the so-called cancellation, when computing a sum in floating point arithmetic, may lead to rather large round-off error. One way to reduce the impact of this destructive phenomenon is to compute the sum in a descending order [2,6]. For the sum $\sum_{j=1}^{n} x_j$, when the summands are all real, the condition number is defined by (see [2])

$$\kappa = \frac{\sum_{j=1}^{n} |x_j|}{\left|\sum_{j=1}^{n} x_j\right|}. \tag{1}$$

Obviously $\kappa \geq 1$. The smaller κ, the less round-off error due to the cancellation.

This definition of the condition number can naturally be generalized to the complex sum $\sum_{j=1}^{n} z_j$:

$$\kappa_{\mathrm{Nat}} = \frac{\sum_{j=1}^{n} |z_j|}{\left|\sum_{j=1}^{n} z_j\right|}. \tag{2}$$

However, in practice, computing a sum is performed by individual summation of real parts and imaginary parts of the summands. Thus, we face the following condition number:

$$\kappa_{\mathrm{Pra}} = max\left\{\frac{\sum_{j=1}^{n}|x_j|}{\left|\sum_{j=1}^{n} x_j\right|}, \frac{\sum_{j=1}^{n}|y_j|}{\left|\sum_{j=1}^{n} y_j\right|}\right\}, \tag{3}$$

where x_j and y_j are the real and imaginary part of z_j, respectively. We call the numbers κ_{Nat} and κ_{Pra} the *natural condition number* and the *practical condition number*, respectively.

Definition 1: Let $-\pi < \theta^{(1)} < \theta^{(2)} \leq \pi$. A $\theta^{(1)} - \theta^{(2)}$ -sector, denoted by $S(\theta^{(1)}, \theta^{(2)})$, is a subset of the complex plane, which defined as follows:

If $\theta^{(2)} - \theta^{(1)} \leq \pi$,

$$S\left(\theta^{(1)}, \theta^{(2)}\right) := \left\{z = r\mathrm{e}^{\mathrm{i}\theta} \mid \theta \in \left(\theta^{(1)}, \theta^{(2)}\right)\right\},$$

and if $\theta^{(2)} - \theta^{(1)} > \pi$,

$$S\left(\theta^{(1)}, \theta^{(2)}\right) := \left\{z = r\mathrm{e}^{\mathrm{i}\theta} \mid \theta \in \left(-\pi, \theta^{(1)}\right) \cup (\theta^{(2)}, \pi]\right\}.$$

The angle of the sector $S\left(\theta^{(1)}, \theta^{(2)}\right)$ is $\theta^{(2)} - \theta^{(1)}$ if $\theta^{(2)} - \theta^{(1)} \leq \pi$, and is $2\pi - \theta^{(2)} + \theta^{(1)}$ if $\theta^{(2)} - \theta^{(1)} > \pi$.

Definition 2: A sector with the angle $\pi/2$ is called a right sector.

Results and discussion

Theorem 1. If $z_j = x_j + iy_j$, $j = 1, \ldots, n$, then

$$\frac{\sqrt{2}}{2} \min\left\{\frac{\sum_{j=1}^{n}|x_j|}{\left|\sum_{j=1}^{n} x_j\right|}, \frac{\sum_{j=1}^{n}|y_j|}{\left|\sum_{j=1}^{n} y_j\right|}\right\} \leq \kappa_{\mathrm{Nat}} \leq \frac{\sum_{j=1}^{n}|x_j|}{\left|\sum_{j=1}^{n} x_j\right|} + \frac{\sum_{j=1}^{n}|y_j|}{\left|\sum_{j=1}^{n} y_j\right|}. \tag{4}$$

In equalities (4) yields the following results.

Corollary 1. If $\kappa_{\mathrm{Pra}} \leq U$, then $\kappa_{\mathrm{Nat}} \leq 2U$.

Corollary 2. If $L \le \kappa_{Nat}$, then $L/2 \le \kappa_{Pra}$.
Corollary 3. If z_j are all located in one quadrant of the complex plane, then $\kappa_{Nat} \le 2$.
The following trivial proposition is comparable to Corollary 3.
Proposition 1. If z_j are all located in one quadrant of the complex plane, then $\kappa_{Pra} = 1$.
Theorem 2. If z_j are all located in a sector $S(\theta^{(1)}, \theta^{(2)})$ with the angle γ less than or equal to $\pi/2$, then

$$\kappa_{Nat} \le \frac{\gamma}{\sin\gamma}.$$

Corollary 4. If z_j are all located in a right sector,

$$\kappa_{Nat} \le \frac{\pi}{2} .$$

Conclusion

In this paper, we have distinguished the natural and the practical condition numbers for computing a complex sum. It has been shown that the angle of the sector (a cone in the complex plane with vertex at the origin) containing the summands plays the key role in the natural condition number: The more acute angle, the smaller natural condition number. Especially, for right-angles we present an upper bound for the natural condition number. For the practical condition number, some results obtained, too: 1) There is no comparative relationship between the natural and the practical condition numbers. 2) The smaller practical condition number, the smaller natural condition number. 3) The larger natural condition number, the larger practical condition number.

Keywords: Filon-Clenshaw-Curtis; Oscillatory integrals; Condition number; Summation.

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