Amenability and Weak Amenability of the Semigroup Algebra $\ell^1(S_T)$

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Abstract

Let $S$ be a semigroup with a left multiplier $T$ on $S$. A new product on $S$ is defined by $T$ related to $S$ and $T$ such that $S$ and the new semigroup $S_T$ have the same underlying set as $S$. It is shown that if $T$ is injective then $\ell^1(S_T) \cong \ell^1(S)\bar{T}$ where, $\bar{T}$ is the extension of $T$ on $\ell^1(S)$. Also, we show that if $T$ is bijective, then $\ell^1(S)$ is amenable if and only if $\ell^1(S_T)$ is so. Moreover, if $S$ completely regular, then $\ell^1(S_T)$ is weakly amenable.

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Introduction

Let $S$ be a semigroup and $T$ a left multiplier on $S$. We present a general method of defining a new product on $S$ which makes $S$ a semigroup. Let $S_T$ denote $S$ with the new product. These two semigroups are sometimes different and we try to find conditions on $S$ and $T$ such that the semigroups $S$ and $S_T$ have the same properties. This idea has started by Birtel in [1] for Banach algebras and continued by Larsen in [11]. Recently, this notion developed by some authors, for more details see [1], [10], [11], [12] and [15]. One of the best result in this work expresses that $L^1(G)\bar{T}$ is Arens regular if and only if $G$ is a compact group [10]. We continue this direction on the regularity of $S$ and $S_T$ and the amenability of their semigroup algebras.

The term of semigroup will be a non-empty set $S$ endowed with an associative binary operation on $S$, defined by $(s, t) \rightarrow st$. If $S$ is also a Hausdorff topological space and the binary operation is jointly continuous, then $S$ is called a topological semigroup.
Let \( p \in S \). Then \( p \) is an idempotent if \( p^2 = p \). The set of all idempotents of \( S \) is denoted by \( E(S) \).

An element \( e \) is a left (right) identity if \( es = s \) (resp. \( se = s \)) for all \( s \in S \). An element \( ecS \) is an identity if it is a left and a right identity. An element \( z \) is a left (resp. right) zero if \( zs = z \) (resp. \( sz = z \)) for all \( s \in S \). An element \( zcS \) is a zero if it is a left and a right zero. We denote any zero of \( S \) by \( 0_S \) (or \( z_S \)). An element \( pcS \) is a regular element of \( S \) if there exists \( tcS \) such that \( p = ptp \) and \( p \) is completely regular if it is regular and \( pt = tp \). We say that \( p \in S \) has an inverse if there exists \( tcS \) such that \( p = pt \) and \( t = tpt \). Note that the inverse of element \( p \in S \) need not be unique. If \( pcS \) has an inverse, then \( p \) is regular and vice versa. Since, if \( p \in S \) is regular, there exists \( s \in S \) such that \( p = psp \). Let \( t = sps \).

So \( p \) has an inverse. We say that \( S \) is a regular (resp. completely regular) semigroup if each \( pcS \) is regular (resp. completely regular). Also \( S \) is an inverse semigroup if each \( p \in S \) has a unique inverse. The map \( T : S \to S \) is called a left (resp. right) multiplier if \( T(st) = T(s)t \) (resp. \( T(st) = sT(t) \) \( (s, tcS) \).

The map \( T : S \to S \) is a multiplier if it is a left and right multiplier. Let \( S \) be a topological semigroup. The net \( (e_\alpha) \subseteq S \) is a left (resp. right) approximate identity if \( \lim_\alpha e_\alpha t = t \) (resp. \( \lim_\alpha t e_\alpha = t \) \( (tcS) \). The net \( (e_\alpha) \subseteq S \) is an approximate identity if it is a left and a right approximate identity.

Let \( S \) be a discrete semigroup. We denote by \( l^1(S) \) the Banach space of all complex function \( f : S \to \mathbb{C} \) having the form

\[
f = \sum_{s \in S} f(s)\delta_s,
\]

such that \( \sum_{s \in S} |f(s)| = \|f\|_1 \) is finite, where \( \delta_s \) is the point mass at \( \{s\} \). For \( f, g \in l^1(S) \) we define the convolution product on \( l^1(S) \) as follow:

\[
f \ast g(s) = \sum_{t_1,t_2 = s} f(t_1)g(t_2) \quad (scS),
\]

with this product \( l^1(S) \) becomes a Banach algebra and is called the semigroup algebra on \( S \).

Remark 1.1. If \( f \in l^1(S) \) then \( f = 0 \) on \( S \) except at most on a countable subset of \( S \). In other words, the set \( A = \{scS : f(s) \neq 0\} \) is at most countable. Since, if \( \{s \in S : |f(s)| \geq \frac{1}{n}\} \), \( A = \bigcup_{n \in \mathbb{N}} A_n \). Set \( \|f\|_1 = M \) and \( n \in \mathbb{N} \) is fixed. Then we have

\[
M = \sum_{s \in S} |f(s)| \geq \sum_{s \in A_n} |f(s)| \geq \sum_{s \in A_n} \frac{1}{n} = \frac{1}{n} |A_n|,
\]
where $|A_n|$ is the cardinality of $A_n$. So $|A_n| \leq nM$. Hence $A_n$ is a finite subset of $S$ and thus $A$ is at most countable.

**Semigroup $S_T$**

Let $T \in \text{Mul}_1(S)$. Then we define a new binary operation "$\circ$" on $S$ as follow:

$$s \circ t = s \cdot T(t) \ (s, t \in S).$$

The set $S$ equip with the new operation "$\circ$" is denoted by $S_T$ and sometimes called "induced semigroup of $S$". Now we have the following results.

**Theorem 2.1.** Let $S$ be a Semigroup. Then (i) if $T \in \text{Mul}_1(S)$ then $S_T$ is a semigroup.

The converse is true if $S$ is left cancellative and $T$ is surjective.

(ii) If $S_T$ is left cancellative and $T$ is surjective, then $T^{-1} \in \text{Mul}_1(S)$.

(iii) If $S$ is a topological semigroup and $S_T$ has a left approximate identity then $T^{-1} \in \text{Mul}_1(S)$.

**Proof.** i) Let $T \in \text{Mul}_1(S)$ and take $r, s, t \in S$. Then

$$r \circ (s \circ t) = r \cdot T(s \cdot t) = r \cdot T(s) \cdot T(t) = (r \cdot T(s)) \cdot T(t) = (r \circ s) \circ t.$$

So, $S_T$ is a semigroup.

Conversely, suppose that $S$ is left cancellative and take $r, s, t \in S$. Since $T$ is surjective, there exists $u \in S$ such that $T(u) = t$. Then

$$rT(st) = rT(sT(u)) = r \circ (s \circ u) = (r \circ s) \circ u = (rT(s))T(u) = r(T(st)).$$

By the left cancellativity of $S$, we have $T(st) = T(s)t \ (r, s \in S)$. So, $T$ is a left multiplier.

ii) We must prove that $T$ is injective. To do this end, take $r, s, u \in S$ and let $T(r) = T(s)$. Then $u \circ r = uT(r) = uT(s) = u \circ s$. So $r = s$, since $S_T$ is left cancellative. Hence $T^{-1}$ exists.

Now, we show that $T^{-1} \in \text{Mul}_1(S)$. Take $r, s \in S$. Then

$$T^{-1}(rs) = T^{-1}[T{T^{-1}(r)s}] = T^{-1}[T(T^{-1}(r)s)] = (T^{-1}T)[T^{-1}(r)s] = T^{-1}(r)s.$$

iii) It is enough to show that $T$ is injective. Take $r, s \in S$ and suppose that $T(r) = T(s)$. Then

$$r = \lim_\alpha e_\alpha \circ r = \lim_\alpha e_\alpha T(r) = \lim_\alpha e_\alpha T(s) = \lim_\alpha e_\alpha \circ s = s.$$
There are many properties that induced from $S$ to semigroup $S_T$. But sometimes they are different.

**Theorem 2.2.** Let $S$ be a Hausdorff topological semigroup and $\text{LeftMult}_1(S)$. If $S$ is commutative then so is $S_T$. The converse is true if $\overline{T(S)} = S$.

**Proof.** Suppose $S$ is commutative and take $r, s \in S$. Then

$$r \circ s = T(s) T(r) = T(s r) = T(r s) = T(r) s = s T(r) = s \circ r.$$  

So, $S_T$ is commutative.

Conversely, Let $S_T$ be commutative and take $r, s \in S$. Then there exist nets $(r_\alpha)$ and $(s_\beta)$ in $S$ such that $\lim_\alpha T(r_\alpha) = r$ and $\lim_\beta T(s_\beta) = s$.

So, we have

$$rs = \lim_\alpha \lim_\beta T(r_\alpha \circ s_\beta) = \lim_\alpha \lim_\beta T(s_\beta \circ r_\alpha) = \lim_\alpha \lim_\beta T(s_\beta) T(r_\alpha) = s r.$$  

Thus $S$ is commutative.

In the sequel, we investigate some relations between two semigroup $S$ and $S_T$ according to the role of the left multiplier $T$.

**Theorem 2.3.** Let $S$ be a semigroup and $\text{LeftMult}_1(S)$. Then

(i) If $T$ is surjective and $S_T$ is an inverse semigroup then $S$ is an inverse semigroup and $T(s^{-1}) = T(s)^{-1}$ for all $s \in S$.

(ii) If $S_T$ is an inverse semigroup and $T$ is injective then $T(S)$ is an inverse subsemigroup of $S$.

(iii) If $T$ is injective then $S_T$ is an inverse semigroup if and only if $S$ is an inverse semigroup.

**Proof.**

i) Suppose that $S_T$ is an inverse semigroup and $T$ is surjective. Define the map $\varphi : S_T \to S$ by $\varphi(s) = T(s)$. Take $r, s \in S$, then

$$\varphi(r \circ s) = T(r \circ s) = T(r) T(s) = \varphi(r) \varphi(s).$$  

So, $\varphi$ is an epimorphism from $S_T$ onto $S$, since $T$ is surjective. By theorem 5.1.4[7], $S$ is an inverse semigroup and $T(s^{-1}) = T(s)^{-1}$ for all $s \in S$.

ii) Suppose that $T$ is injective and $S_T$ is an inverse semigroup. Evidently, $T(S)$ is a subsemigroup of $S$. We show that it is an inverse semigroup. Take $s \in T(S)$. There exists $t \in S$ such that $s = T(t)$. Also, there exists a unique element $u \in S$ such that $t = u \circ T(t)$, since $S_T$ is an inverse semigroup. Therefore, $T(t) = T(t) T(u) T(t)$, or $s = s \circ T(u) \circ s$. Of course, $T(u)$ is unique because $u \in S$ is unique and $T$ is injective. Hence $T(S)$ is an inverse subsemigroup of $S$. 


iii) Suppose that $T$ is bijective and let $S_T$ be an inverse semigroup. Since $T$ is injective and surjective, by (i) and (ii), $S = T(S)$ is an inverse semigroup.

Conversely, suppose that $S$ is an inverse semigroup. Since $T$ is bijective, by theorem 2.1(ii), $T^{-1} \in \text{Mul}_S(S)$. So $\varphi^{-1}: S \rightarrow S_T$ defined by $\varphi^{-1}(s) = T^{-1}(s)$ is an epimorphism. Hence by (i) $S_T$ is an inverse semigroup.

We say that $T \in \text{Mul}_S(S)$ is an inner left multiplier if it has the form $T = L_s$ for some $s \in S$ where $L_s(t) = st \quad (teS)$.

If $T \in \text{Mul}_S(S)$ is inner, then each ideal of $S$ is permanent under $T$; that is $T(I) \subseteq I$ for all ideal $I$ of $S$. It is easily to see that if $S$ has an identity, then each $T \in \text{Mul}_S(S)$ is inner.

Let $S$ be a semigroup. Then $S$ is called semisimple if $I^2 = I$ for all ideal $I$ of $S$ (see [9], page 95 for more details).

**Theorem 2.4.** Let $S$ be a semigroup with an identity and $T \in \text{Mul}_S(S)$. If $S_T$ is semisimple, then $S$ is so. The converse is true if $S_T$ is left cancellative and $T$ is surjective.

**Proof.** Since $S$ is unital there exists $\mu \in S$ such that $T = L_{\mu}$. Suppose that $S_T$ is semisimple and $I$ is an ideal of $S$. Then

$$I \circ S = I \circ T(S) \subseteq I S \subseteq I.$$

Similarly, $S \circ I \subseteq I$. It follows that $I$ is an ideal of $S_T$. By the hypothesis $(I_T)^2 = I \circ I = I$. Now, take $r \in I$ then there are $s, t \in I$ such that

$$r = s \circ t = st(t) = s(\mu t) \in I^2.$$

So we show that $I^2 = I$ and hence $S$ is semisimple.

Conversely, assume that $S_T$ is left cancellative and $T \in \text{Mul}_S(S)$ is surjective then by theorem 2.1(ii), $T^{-1} \in \text{Mul}_S(S)$. So, there exists $b \in S$ such that $T^{-1} = L_b$. Suppose that $S_T = S_{T^{-1}}$. Then we have.

$$S = S_{T^{-1}} = (S_{T})_{T^{-1}} = S_{T^{-1}}.$$

By hypothesis and above the proof, $S_T = S_{T^{-1}}$ is semisimple.

**Semigroup Algebra** $L^1(S_T)$

We say that a discrete semigroup $S$ is amenable if there exists a positive linear functional on $L^\alpha(S)$ called a mean such that $m(1) = 1$ and $m(l_s f) = m(f)$, $m(r_s f) = m(f)$ for each $s \in S$, where $l_s f(t) = f(st)$ and $r_s f(t) = f(ts)$ for all $t \in S$. The definition of amenable group is similar to semigroup case. Refer to [12] for more details.
Let $\mathfrak{A}$ be a Banach algebra and let $X$ be a Banach $\mathfrak{A}$–bimodule. A derivation from $\mathfrak{A}$ to $X$ is a linear map $D: \mathfrak{A} \to X$ such that
$$D(ab) = D(a) \cdot b + a \cdot D(b) \quad (a, b \in \mathfrak{A}).$$
A derivation $D$ is inner if there exists $x \in X$ such that
$$D(a) = a \cdot x - x \cdot a \quad (a \in \mathfrak{A}).$$

The Banach algebra $\mathfrak{A}$ is amenable if every bounded derivation $D: \mathfrak{A} \to X^*$ is inner for all Banach $\mathfrak{A}$–bimodule $X$. Where $X^*$ is the dual space of $X$. We say that the Banach algebra $\mathfrak{A}$ is weakly amenable if any bounded derivation $D$ from $\mathfrak{A}$ to $\mathfrak{A}^*$ is inner. For more details see [12], [16].

If $S$ is a commutative semigroup, by theorem 5.8 of [8] $\ell^1(S)$ is called semisimple if and only if for all $x, y \in S$, $x^2 = y^2 = xy$ implies $x = y$.

**Theorem 3.1.** Let $S$ be a commutative semigroup and let $T \in \text{Mul}_l(S)$ be injective. Then $\ell^1(S)$ is semisimple if and only if $\ell^1(S_T)$ is semisimple.

**Proof.** Take $r, s \in S$. Then $r^2 = s^2 = rs$ if and only if $T(r^2) = T(s^2) = T(r)T(s)$ or equivalently $r \circ r = s \circ s = r \circ s$, because $T$ is injective. So, by theorem 5.8 [8], $\ell^1(S)$ is semisimple if and only if $\ell^1(S_T)$ is semisimple.

**Theorem 3.2.** Let $S$ be a discrete semigroup and $T \in \text{Mul}_l(S)$. Then (i) The left multiplier $T$ has an extension $\tilde{T} \in \text{Mul}_l(\ell^1(S))$ with the norm decreasing.

(ii) The left multiplier $T$ is injective if and only if so is $\tilde{T}$.

(iii) If $T$ is injective then $\tilde{T}$ is an isometry and also $\ell^1(S_T)$ and $\left(\ell^1(S)\right)_T$ are isomorphic.

**Proof.** (i) An arbitrary element $f \in \ell^1(S)$ is of the form $f: S \to \mathbb{C}$ such that $f(x) = 0$ except at the most countable subset $A$ of $S$. If $A$ is a finite subset of $S$ then $f = \sum_{k=1}^n f(x_k) \delta_{x_k}$ for some fixed $n \in \mathbb{N}$. So in general we have
$$f = \sum_{x \in S} f(x) \delta_x = \sum_{x \in A} f(x) \delta_x = \sum_{k=1}^n f(x_k) \delta_{x_k}.$$  

Now, for each $n \in \mathbb{N}$, let $f_n = \sum_{k=1}^n f(x_k) \delta_{x_k}$ and define $\tilde{T}: \ell^1(S) \to \ell^1(S)$ by
$$\tilde{T}(\delta_x) = \delta_{T(x)} \quad (x \in S),
\tilde{T}(f_n) = \sum_{k=1}^n f(x_k) \delta_{x_k} = \tilde{f}_n.$$  

For each $m, n \in \mathbb{N}$ where $n \geq m$, we have
$$\|\tilde{T}(f_n) - \tilde{T}(f_m)\|_1 = \|\tilde{f}_n - \tilde{f}_m\|_1 = \|\sum_{k=m}^n f(x_k) \tilde{T}(\delta_{x_k})\|_1 = \|\sum_{k=m}^n f(x_k) \tilde{T}(\delta_{T(x_k)})\|_1 \\ \leq \sum_{k=m}^n |f(x_k)| = \|f_n - f_m\|_1.$$
So \( \{\tilde{T}(f_n)\}_n \) is a Cauchy sequence and it is convergent. Now, we define \( \tilde{T}(f) = \lim_n \tilde{T}_n \). Then the definition is well defined. Hence
\[
\tilde{T}(f) = \sum_{k=1}^{\infty} f(x_k) \delta_{x_k} = \tilde{f} ,
\]
also
\[
\|\tilde{f}\|_1 \leq \sum_{x \in A} |f(x_k)| = \|f\|_1 \quad \text{or} \quad \|\tilde{T}(f)\|_1 \leq \|f\|_1 .
\]
It shows that \( \tilde{T} \) is norm decreasing.

In the following, we extend \( \tilde{T} \) by linearity. Let \( f, g \in \ell^1(S) \). Then there are two at most countable sub set \( A, B \) of \( S \) such that
\[
f = \sum_{x \in A} f(x) \delta_x , \quad g = \sum_{x \in B} g(x) \delta_x .
\]
Suppose that \( D = A \cup B \). So we have \( f + g = \sum_{x \in D} (f(x) + g(x)) \delta_x \).
Then, it follows that
\[
\tilde{T}(f + g) = \tilde{T} + \tilde{g} = \sum_{x \in D} (f(x) + g(x)) \delta_x = \sum_{x \in A} f(x) \tilde{T}(\delta_x) + \sum_{x \in B} g(x) \tilde{T}(\delta_x) = \tilde{f} + \tilde{g} .
\]
Also, if \( \alpha \in \mathbb{C} \), we have
\[
\tilde{T}(\alpha f) = \alpha \tilde{T} f = \sum_{x \in A} \alpha f(x) \tilde{T}(\delta_x) = \alpha \sum_{x \in A} f(s) \tilde{T}(\delta_x) = \alpha \tilde{T}(f) .
\]
Therefore, \( \tilde{T} \) is a bounded linear isometry.

Now, we prove that \( \tilde{T} \in \text{Mul}_1(\ell^1(S)) \). Take \( x, y \in S \). Then
\[
\tilde{T}(\delta_x \ast \delta_y) = \tilde{T}(\delta_{xy}) = \delta_{T(xy)} = \delta_{T(x) \ast T(y)} = \delta_{T(x) \ast T(y)} = \tilde{T}(\delta_x) \ast \delta_y = \tilde{T}(\delta_x) \ast \delta_y .
\]
Let \( y \in S \) be fixed and \( f, g \in \ell^1(S) \). Then
\[
\tilde{T}(f \ast \delta_y) = \tilde{T}(\sum_{x \in A} f(x) \delta_{xy}) = \sum_{x \in A} f(x) \tilde{T}(\delta_{xy}) = \left( \sum_{x \in A} \tilde{T}(\delta_x) \right) \ast \delta_y = \tilde{T}(f) \ast \delta_y = \tilde{T}(f) \ast \delta_y .
\]
In the general case, we have
\[
\tilde{T}(f * g) = \tilde{T}(\sum_{x \in A} f(x) (\sum_{y \in B} g(y)) \delta_{xy}) = \sum_{x \in A} f(x) \sum_{y \in B} g(y) \tilde{T}(\delta_x) \ast \delta_y = \sum_{x \in A} f(x) \tilde{T}(\delta_x) \ast \sum_{y \in B} g(y) \delta_y = \tilde{T}(f) \ast g .
\]
This shows that \( \tilde{T} \) is a multiplier on \( \ell^1(S) \).

(ii) Let \( T \) be injective. Take \( x, y \in S \) and suppose that \( \tilde{T}(\delta_x) = \tilde{T}(\delta_y) \). Then \( \delta_{T(x)} = \tilde{T}(\delta_x) = \tilde{T}(\delta_y) = \delta_{T(y)} \).
Therefore, \( T(x) = T(y) \). Since \( T \) is injective, we have \( x = y \). It follows that \( \delta_x = \delta_y \), consequently \( \tilde{T} \) is injective.

Conversely, the same argument shows that the converse holds.

(iii) Let \( T \) be injective and \( f \in \ell^1(S) \). Then there exists at most a countable subset \( A \subseteq S \) such that
Since $A$ and $T(A)$ have the same cardinal number, $\|T(f)\| = \|\sum_{x \in A} f(x) \delta_x\| = \sum_{x \in A} |f(x)|\|\delta_x\|_1$, so $T$ is an isometry.

Now, we can define a new multiplication "\(\boxplus\)" on $\ell^1(S)$ as follow

$$f \boxplus g = f \ast Tg \quad (f, g \in \ell^1(S)).$$

By a similar argument in theorem1.31 [10], $\ell^1(S)$ with the new product is a Banach algebra that is denoted it by $\ell^1(S)_{\tilde{T}}$. We define the map $\Psi: \ell^1(S_T) \to \ell^1(S)_{\tilde{T}}$, by

$$\Psi(\delta_x) = \delta_{\tilde{x}} \quad (x \in S).$$

Take $x, y \in S$. Then

$$\Psi(\delta_x \ast \delta_y) = \Psi(\delta_{x \ast y}) = \delta_{xT(y)} = \delta_x \ast \delta_{T(y)}$$

$$= \delta_x \ast \tilde{T}(\delta_y) = \delta_x \boxplus \delta_y$$

$$= \Psi(\delta_x) \boxplus \Psi(\delta_y).$$

So, in general case, we have

$$\Psi(f \ast g) = \Psi(f) \boxplus \Psi(g) \quad (f, g \in \ell^1(S)).$$

Thus, $\Psi$ is an isomorphism. Therefore $\ell^1(S_T)$ and $\ell^1(S)_{\tilde{T}}$ are isomorphic.

**Theorem 3.3.** Let $S$ be a semigroup and $T \in Mul(S)$ be bijective. Then $\ell^1(S)$ is amenable if and only if $\ell^1(S_T)$ is amenable.

**Proof.** By theorem 3.2, we have $\ell^1(S_T) \cong \ell^1(S)_{\tilde{T}}$. Suppose that $\ell^1(S_T)$ is amenable and define $\varphi: \ell^1(S_T) \to \ell^1(S)$ by $\varphi(f) = \tilde{T}(f)$. Take $x, y \in S$. Then

$$\varphi(\delta_x \boxplus \delta_y) = \tilde{T}(\delta_x \boxplus \delta_y) = \tilde{T}(\delta_{xT(y)}) = \tilde{T}(\delta_x \ast \delta_{T(y)}) = \tilde{T}(\delta_x) \ast \delta_{T(y)}$$

$$= \tilde{T}(\delta_x) \ast \tilde{T}(\delta_y) = \varphi(\delta_x) \ast \varphi(\delta_y).$$

Now, by induction and continuity of $\tilde{T}$, we have

$$\varphi(f \boxplus g) = \varphi(f) \ast \varphi(g).$$

If $T$ is bijective, $\tilde{T}$ is bijective. Therefore $\varphi$ is an epimorphism of $\ell^1(S_T)$ onto $\ell^1(S)$.

Hence, by proposition 2.3.1 [16], $\ell^1(S)$ is amenable.

Conversely, suppose that $\ell^1(S)$ is amenable. Since $T$ is bijective, $\tilde{T}$ is bijective. Therefore $\tilde{T}^{-1}$ exists. Now define $\theta: \ell^1(S) \to \ell^1(S_T) \cong \ell^1(S)_{\tilde{T}}$ by $\theta(f) = \tilde{T}^{-1}(f)$.

Take $x, y \in S$. Then

$$\theta(\delta_x \ast \delta_y) = \tilde{T}^{-1}(\delta_{xy}) = \tilde{T}^{-1}(\delta_x)\tilde{T}^{-1}(\delta_y) = \tilde{T}^{-1}(\delta_x) \boxplus \tilde{T}^{-1}(\delta_y)$$

$$= \theta(\delta_x) \boxplus \theta(\delta_y).$$
Similarly $\theta$ is an epimorphism from $\ell^1(S)$ onto $\ell^1(S_T)$. By proposition 2.3.1 [16] $\ell^1(S_T)$ is amenable.

Note that, in general, it is not known when $\ell^1(S)$ is weakly amenable. For more details see [2].

**Theorem 3.4.** Let $S$ be a semigroup and $T \in \text{Mul}_l(S)$ be bijective. Then, if $S$ is completely regular then $\ell^1(S_T)$ is weakly amenable.

**Proof.** It is enough to prove that $S_T$ is completely regular, then by theorem 3.6 [2], $\ell^1(S_T)$ can be weakly amenable. Take $s \in S$. Then there exists $r \in S$ such that $T(s) = T(s)T(r)T(s)$, $T(r)T(s) = T(s)T(r)$, since $T$ is bijective and $S = T(S)$ is completely regular. So we have $T(s) = T(s \circ r \circ s)$ and $T(r \circ s) = T(s \circ r)$. Hence $s = s \circ r \circ s$ and $r \circ s = s \circ r$ for some $r \in S$, since $T$ is injective. Therefore $S_T$ is completely regular.

**Corollary 3.5.** Suppose that $S$ is a commutative completely regular semigroup and $T \in \text{Mul}_l(S)$ is injective. Then $\ell^1(T(S)_T)$ is weakly amenable.

**Proof.** [2, theorem 3.6] $\ell^1(S)$ is weakly amenable. Define $\varphi: S \to \ell^1(S)_T$ by

$$\varphi(s) = T^{-1}(s) \quad (s \in S).$$

We show that $\varphi$ is a homomorphism. Take $s \in S$, then we have

$$\varphi(rs) = T^{-1}(rs) = T^{-1}(r)s = T^{-1}(r) \circ (T^{-1}s).$$

So $\varphi$ is a homomorphism. Then by proposition 2.1[7], $\ell^1(T(S)_T)$ is weakly amenable.

In the case that $S$ is a group, it is easy to see that the amenability of $S$ implies the amenability of $\ell^1(S_T)$. Indeed, when $S$ is a group, by theorem 2.1, $S_T$ is a semigroup and one can easily prove that $S_T$ is also a group. On the other hand, $\text{Mul}_l(S) \cong S$ because $S$ is a unital semigroup, so each $T \in \text{Mul}_l(S)$ is inner and of the form $T = L_s$ for some $s \in S$. Also $T^{-1} = L_{s^{-1}}$ exists, since $S$ is a group. Then the map $\theta: S_T \to S$ defined by $\theta(s) = T(s)$ is an isomorphism; that is $S \cong S_T$. Thus we have the following result:

**Corollary 3.6.** Let $S$ be a cancellative regular discrete semigroup. Then $\ell^1(S)$ is amenable if and only if $\ell^1(S_T)$ is amenable.

**Proof.** By [9, Exercise 2.6.11] $S$ is a group. So the assertion holds by [15, theorem 2.1.8]
Examples

In this section we present some examples which either comment on our results or indicate necessary condition in our theorems.

4.1. There are semigroups $S$ and $T \in Mult(S)$ such that the background semigroups $S$ are not commutative but their induced semigroups $S_T$ are commutative.

This example shows that the condition $\overline{T(S)} = S$, in theorem 2.2, can not be omitted.

Let $S$ be the set $\{a, b, c, d, e\}$ with operation table given by

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Clearly $(S, \cdot)$ is a non-commutative semigroup. Now, put $T = L_a$ where $L_a(x) = ax$ for all $x \in S$. One can get easily the operation table of $S_T$ as follow:

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</table>

The operation table shows that the induced semigroup $S_T$ is commutative and $T(S) \neq S$. Also the other induced semigroup $S_T$ is commutative for $T = L_d$ analogously.

Now we present some important theorems from[14] that we need in the following examples:

Theorem 4.2. Let $S$ be a semigroup. Suppose that $t^1(S)$ is amenable. Then

(i) $S$ is amenable
(ii) $S$ is regular.
(iii) $E(S)$ is finite.
(iv) $t^1(S)$ has an identity.

Proof. (i) That is lemma 3 in [5].

(ii) and (iii) See theorem 2 in[6].
(iv) That is corollary 10.6 in[4].

**Theorem 4.3.** Let $S$ be a finite semigroup. Then the following statements are equivalent:

(i) $\ell^1(S)$ is amenable.
(ii) $S$ is regular and $\ell^1(S)$ is nuiital.
(ii) $S$ is regular and $\ell^1(S)$ is semisimple.

**Proof.** Refer to [3].

4.4. There are semigroups $S$ and $T \in \text{Mul}(S)$ such that $S$ and $\ell^1(S)$ are amenable but $S_T$ is not regular and also, $\ell^1(S_T)$ is not amenable. This example shows that two semigroup algebras $\ell^1(S)$ and $\ell^1(S_T)$ can be different in some properties. Also, it notifies that the bijectivity of $T$ in the theorem 3.3 is essential. Put $S = \{x_0, x_1, x_2, \ldots, x_n\}$ with the operation $x_i x_j = x_{\text{Max}(i,j)}$ ($0 \leq i, j \leq n$, $n \geq 2$).

Then $S$ is a semigroup. Since

$$\text{Max}\{i, \text{Max}(j,k)\} = \text{Max}\{\text{Max}(i,j), k\} = \text{Max}(i,j,k).$$

We denote it by $S_v$. This semigroup is commutative. So by (0.18) in [12], it is amenable. $S_v$ is a unital semigroup and has a zero; indeed, $e_s = x_0$ and $a_s = x_n$. Also, it is a regular semigroup and $\text{Mul}(S_v) \cong S_v$ because $S_v$ has an identity.

Evidently, $S_v$ is regular since each $s \in S_v$ is idempotent. The semigroup algebra $\ell^1(S_v)$ is a unital algebra because $S_v$ has an identity. So by theorem 4.3 (ii) $\ell^1(S_v)$ is amenable.

Now, take $T = L_{x_k}$ for a fixed $x_k \in S$ where $k \geq 1$. By theorem 2.2, $(S_v)_T$ is commutative so is amenable. We show that $T$ is neither injective and nor surjective. Take $x_i \in S_v$, then $T x_i = x_k x_i = x_{\text{Max}(k,i)}$. So

$$T(S_v) = \{x_k, x_{k+1}, \ldots, x_n\} \neq S_v.$$

Hence, $T$ is not surjective.

Again, take distinct elements $x_i, x_j$ in $S_v$ for some $i, j < k$ such that $T(x_i) = T(x_j)$. Then we have $x_{\text{Max}(k,i)} = x_{\text{Max}(k,j)}$ but $x_i \neq x_j$. So $T$ is not injective.

We prove that $(S_v)_T$ is not regular. If $(S_v)_T$ is regular, then for $x_k \in S_v$ there exists an element $x_j \in S_v$ such that

$$x_{k-1} = x_{k-1} \circ x_j \circ x_{k-1} = x_{\text{Max}(k,j)}.$$
That implies that $\max\{k, j\} = k - 1$; which is impossible. Consequently, by theorem 4.2 (ii) or 4.3 (ii), $\ell^1((S_v)_T)$ is not amenable.

Also, the inequality $S_v \circ S_v = \{x_k, x_{k+1}, \ldots, x_n\} \neq S_v$ shows that $\ell^1((S_v)_T)$ is not weakly amenable. In the next example we show that in the theorem 3.2 (iii) the condition "injectivity of" can not be omitted.

**4.5** There are a semigroup $S$ and $T \in Mul_{l}(S)$ such that $T \in Mul_{l}(S)$ is not injective and the corresponding $\tilde{T} \in Mul_{l}(\ell^1(S_T))$ is not an isometry.

Suppose that $S_v$ is a semigroup as in example 4.4 and $T = L_{x_k}$ for some fixed $1 < k < n$. If $f \in \ell^1(S_v)$ then $f = \sum_{i=0}^{n} f(x_i) \delta_{x_i}$ and also $\tilde{T}(f) = \sum_{i=0}^{n} f(x_i) \delta_{T(x_i)}$. But

$$T(x_i) = \begin{cases} x_i & k < i \leq n \\ x_k & 0 \leq i \leq k \end{cases},$$

so

$$\tilde{T}(f) = \left( \sum_{i=0}^{k} f(x_i) \right) \delta_{x_k} + \sum_{i=k+1}^{n} f(x_i) \delta_{T(x_i)}.$$ 

Hence

$$\|\tilde{T}(f)\| = \left| \sum_{i=0}^{k} f(x_i) \right| + \sum_{i=k+1}^{n} |f(x_i)| \leq \sum_{i=0}^{k} |f(x_i)| + \sum_{i=k+1}^{n} |f(x_i)| = \|f\|_1.$$ 

It shows that $\tilde{T}$ is not an isometry.

**4.6.** There are semigroups $S$ and $T \in Mul_{l}(S)$ such that $\ell^1(S)$ is semisimple. But $\ell^1(S_T)$ is not semisimple. This example remind that, in theorem 3.1 the multiplier $T$ must be injective.

Let $S$ be a set $\{x_0, x_1, \ldots, x_n\}$ where $n \in \mathbb{N}$ and $n \geq 3$ is fixed. by operation given by $xy = x_{\min\{i, j\}}$, $S$ is a commutative semigroup. Since

$$\min\{i, \min\{j, k\}\} = \min\{\min\{i, j\}, k\} = \min\{i, j, k\} \quad (i, j, k \in \mathbb{N}).$$

We denote it briefly by $S_{\wedge}$. For each $x, y \in S$ the equality $x^2 = y^2 = xy$ implies $x = y$. So by Theorem 5.8 [8] $\ell^1(S_{\wedge})$ is semisimple.

Now, let $T = L_{x_k}$ for a fixed $1 \leq k < n - 1$. It is easy to see that $T(x_k) = T(x_n)$ but $x_k \neq x_n$. So the multiplier $T$ is not injective.

We show that neither $S_{\wedge}$ nor $\ell^1(S_{\wedge})_T$ is semisimple.

Each ideal of $S$ is of the form

$$I_m = \{x_0, x_1, \ldots, x_m\} \quad (m \leq n).$$

We claim that $S_T$ is not semisimple. Since for each $m \in \mathbb{N}$ we have
On the other hand, for each $x_i, x_j \in S$ where $i \neq j$ and $i, j > k$, we have $x_i \circ x_i = x_j \circ x_j = x_i \circ x_j = x_k$, while $x_i \neq x_j$. Thus, Theorem 5.8 [8] shows that $\ell^1(S, \alpha)$ is not semisimple.

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Reference


