Amenability and Weak Amenability of the Semigroup Algebra $\ell^1(S_T)$

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Abstract

Let $S$ be a semigroup with a left multiplier $T$ on $S$. A new product on $S$ is defined by $T$ related to $S$ and $T$ such that $S$ and the new semigroup $S_T$ have the same underlying set as $S$. It is shown that if $T$ is injective then $\ell^1(S_T) \cong \ell^1(S)\bar{T}$ where, $\bar{T}$ is the extension of $T$ on $\ell^1(S)$. Also, we show that if $T$ is bijective, then $\ell^1(S)$ is amenable if and only if $\ell^1(S_T)$ is so. Moreover, if $S$ completely regular, then $\ell^1(S_T)$ is weakly amenable.

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Introduction

Let $S$ be a semigroup and $T$ be a left multiplier on $S$. We present a general method of defining a new product on $S$ which makes $S$ a semigroup. Let $S_T$ denote $S$ with the new product. These two semigroups are sometimes different and we try to find conditions on $S$ and $T$ such that the semigroups $S$ and $S_T$ have the same properties. This idea has started by Birtel in [1] for Banach algebras and continued by Larsen in [11]. Recently, this notion developed by some authors, for more details see [1], [10], [11], [12] and [15]. One of the best result in this work expresses that $L^1(G)_T$ is Arens regular if and only if $G$ is a compact group [10]. We continue this direction on the regularity of $S$ and $S_T$ and the amenability of their semigroup algebras.

The term of semigroup will be a non-empty set $S$ endowed with an associative binary operation on $S$, defined by $(s, t) \rightarrow st$. If $S$ is also a Hausdorff topological space and the binary operation is jointly continuous, then $S$ is called a topological semigroup.
Let $p \in S$. Then $p$ is an idempotent if $p^2 = p$. The set of all idempotents of $S$ is denoted by $E(S)$.

An element $e$ is a left (right) identity if $es = s$ (resp. $se = s$) for all $s \in S$. An element $eS$ is an identity if it is a left and a right identity. An element $z$ is a left (resp. right) zero if $zs = z$ (resp. $sz = z$) for all $s \in S$. An element $zeS$ is a zero if it is a left and a right zero. We denote any zero of $S$ by $0_S$ (or $z_S$). An element $pcS$ is a regular element of $S$ if there exists $tcS$ such that $p = ptp$ and $p$ is completely regular if it is regular and $pt = tp$. We say that $p \in S$ has an inverse if there exists $tcS$ such that $p = ptp$ and $t = tpt$. Note that the inverse of element $p \in S$ need not be unique. If $pcS$ has an inverse, then $p$ is regular and vice versa. Since, if $p \in S$ is regular, there exists $s \in S$ such that $p = psp$. Let $t = sps$. Then $p = psp = (p(sp)s)p = ptp$, $t = sps = s(psp)s = (spsp)p(sps) = tpt$.

So $p$ has an inverse. We say that $S$ is a regular (resp. completely regular) semigroup if each $pcS$ is regular (resp. completely regular). Also $S$ is an inverse semigroup if each $p \in S$ has a unique inverse. The map $T : S \rightarrow S$ is called a left (resp. right) multiplier if

$$T(st) = T(s)t \quad (\text{resp. } T(st) = sT(t)) \quad (s, tcS).$$

The map $T : S \rightarrow S$ is a multiplier if it is a left and right multiplier. Let $S$ be a topological semigroup. The net $(e_\alpha) \subseteq S$ is a left (resp. right) approximate identity if $\lim_{\alpha} e_\alpha t = t$ (resp. $\lim_{\alpha} te_\alpha = t$) $(tcS)$. The net $(e_\alpha) \subseteq S$ is an approximate identity if it is a left and a right approximate identity.

Let $S$ be a discrete semigroup. We denote by $l^1(S)$ the Banach space of all complex function $f : S \rightarrow \mathbb{C}$ having the form

$$f = \sum_{s \in S} f(s)\delta_s,$$

such that $\sum_{s \in S}|f(s)| = ||f||_1$ is finite, where $\delta_s$ is the point mass at $\{s\}$. For $f, g \in l^1(S)$ we define the convolution product on $l^1(S)$ as follow:

$$f * g(s) = \sum_{t_1, t_2 = s} f(t_1)g(t_2) \quad (scS),$$

with this product $l^1(S)$ becomes a Banach algebra and is called the semigroup algebra on $S$.

Remark 1.1. If $f \in l^1(S)$ then $f = 0$ on $S$ except at most on a countable subset of $S$. In other words, the set $A = \{scS : f(s) \neq 0\}$ is at most countable. Since, if $A_n = \{scS : |f(s)| \geq \frac{1}{n}\}$, $A = \bigcup_{n \in \mathbb{N}}A_n$. Set $||f||_1 = M$ and $n \in \mathbb{N}$ is fixed. Then we have

$$M = \sum_{s \in S} |f(s)| \geq \sum_{s \in A_n} |f(s)| \geq \sum_{s \in A_n} \frac{1}{n} = \frac{1}{n} |A_n|,$$
where $|A_n|$ is the cardinality of $A_n$. So $|A_n| \leq nM$. Hence $A_n$ is a finite subset of $S$ and thus $A$ is at most countable.

**Semigroup $S_T$**

Let $T \in \text{Mul}_1(S)$. Then we define a new binary operation "$\circ$" on $S$ as follow:

$$s \circ t = s \circ T(t) = (s, t \in S).$$

The set $S$ equiped with the new operation "$\circ$" is denoted by $S_T$ and sometimes called "induced semigroup of $S$". Now we have the following results.

**Theorem 2.1.** Let $S$ be a Semigroup. Then (i) if $T \in \text{Mul}_1(S)$ then $S_T$ is a semigroup.

The converse is true if $S$ is left cancellative and $T$ is surjective.

(ii) If $S_T$ is left cancellative and $T$ is surjective, then $T^{-1} \in \text{Mul}_1(S)$.

(iii) If $S$ is a topological semigroup and $S_T$ has a left approximate identity then $T^{-1} \in \text{Mul}_1(S)$.

**Proof.**

i) Let $Te\text{Mul}_1(S)$ and take $r,s,t \in S$. Then

$$r \circ (s \circ t) = r \circ T(s \circ t) = r \circ T(s \circ T(t)) = r \circ T(s) \circ T(t) = (r \circ s) \circ t$$

So, $S_T$ is a semigroup.

Conversely, suppose that $S$ is left cancellative and take $r,s,t \in S$. Since $T$ is surjective, there exists $u \in S$ such that $T(u) = t$. Then

$$rT(st) = rT(sT(u)) = r \circ (s \circ u) = (r \circ s) \circ u = (rT(s))T(u) = rT(st).$$

By the left cancellativity of $S$, we have $T(st) = T(s)T(t) = (r,s \in S)$. So, $T$ is a left multiplier.

ii) We must prove that $T$ is injective. To do this end, take $r,s,u \in S$ and let $T(r) = T(s)$.

Then $u \circ r = uT(r) = uT(s) = u \circ s$. So $r = s$, since $S_T$ is left cancellative. Hence $T^{-1}$ exists.

Now, we show that $T^{-1} \in \text{Mul}_1(S)$. Take $r,s \in S$. Then

$$T^{-1}(rs) = T^{-1}[TT^{-1}(r)s] = T^{-1}[T(T^{-1}(r)s)] = (T^{-1}T)[T^{-1}(r)s] = T^{-1}(r)s.$$  

iii) It is enough to show that $T$ is injective. Take $r,s \in S$ and suppose that $T(r) = T(s)$.

Then

$$r = \lim_{\alpha} e_{\alpha} \circ r = \lim_{\alpha} e_{\alpha} T(r) = \lim_{\alpha} e_{\alpha} T(s) = \lim_{\alpha} e_{\alpha} \circ s = s.$$
There are many properties that induced from $S$ to semigroup $S_T$. But sometimes they are different.

**Theorem 2.2.** Let $S$ be a Hausdorff topological semigroup and $\text{TeMul}_1(S)$. If $S$ is commutative then so is $S_T$. The converse is true if $\overline{T(S)} = S$.

**Proof.** Suppose $S$ is commutative and take $r, s \in S$. Then
$$r \circ s = r T(s) = T(s)r = T(sr) = T(rs) = T(r) s = sT(r) = s \circ r .$$
So, $S_T$ is commutative.

Conversely, Let $S_T$ be commutative and take $r, s \in S$. Then there exist nets $(r_\alpha)$ and $(s_\beta)$ in $S$ such that $\lim_\alpha T(r_\alpha) = r$ and $\lim_\beta T(s_\beta) = s$.

So, we have
$$rs = \lim_\alpha \lim_\beta T(r_\alpha \circ s_\beta) = \lim_\beta \lim_\alpha T(s_\beta \circ r_\alpha) = \lim_\alpha \lim_\beta T(s_\beta) T(r_\alpha) = s \circ r .$$
Thus $S$ is commutative.

In the sequel, we investigate some relations between two semigroup $S$ and $S_T$ according to the role of the left multiplier $T$.

**Theorem 2.3.** Let $S$ be a semigroup and $\text{TeMul}_1(S)$. Then

(i) If $T$ is surjective and $S_T$ is an inverse semigroup then $S$ is an inverse semigroup and $T(s^{-1}) = T(s)^{-1}$ for all $s \in S$.

(ii) If $S_T$ is an inverse semigroup and $T$ is injective then $T(S)$ is an inverse subsemigroup of $S$.

(iii) If $T$ is bijective then $S_T$ is an inverse semigroup if and only if $S$ is an inverse semigroup.

**Proof.** i) Suppose that $S_T$ is an inverse semigroup and $T$ is surjective. Define the map $\varphi: S_T \to S$ by $\varphi(s) = T(s)$. Take $r, s \in S$, then
$$\varphi(r \circ s) = T(r \circ s) = T(r) T(s) = \varphi(r) \varphi(s) .$$
So, $\varphi$ is an epimorphism from $S_T$ onto $S$, since $T$ is surjective. By theorem 5.1.4[7], $S$ is an inverse semigroup and $T(s^{-1}) = T(s)^{-1}$ for all $s \in S$.

ii) Suppose that $T$ is injective and $S_T$ is an inverse semigroup. Evidently, $T(S)$ is a subsemigroup of $S$. We show that it is an inverse semigroup. Take $s \in T(S)$. There exists $t \in S$ such that $s = T(t)$. Also, there exists a unique element $u \in S$ such that $t = u \circ t$, since $S_T$ is an inverse semigroup. Therefore, $T(t) = T(t) T(u) T(t)$, or $s = s \circ T(u) \circ s$.

Of course, $T(u)$ is unique because $u \in S$ is unique and $T$ is injective. Hence $T(S)$ is an inverse subsemigroup of $S$.  


iii) Suppose that $T$ is bijective and let $S_T$ be an inverse semigroup. Since $T$ is injective and surjective, by (i) and (ii), $S = T(S)$ is an inverse semigroup.

Conversely, suppose that $S$ is an inverse semigroup. Since $T$ is bijective, by theorem 2.1(ii), $T^{-1} \in \text{Mul}_l(S)$. So $\varphi^{-1}: S \to S_T$ defined by $\varphi^{-1}(s) = T^{-1}(s)$ is an epimorphism. Hence by (i) $S_T$ is an inverse semigroup.

We say that $T \in \text{Mul}_l(S)$ is an inner left multiplier if it has the form $T = L_s$ for some $s \in S$ where $L_s(t) = st$ $(teS)$.

If $T \in \text{Mul}_l(S)$ is inner, then each ideal of $S$ is permanent under $T$; that is $T(I) \subseteq I$ for all ideal $I$ of $S$. It is easily to see that if $S$ has an identity, then each $T \in \text{Mul}_l(S)$ is inner.

Let $S$ be a semigroup. Then $S$ is called semisimple if $I^2 = I$ for all ideal $I$ of $S$ (see [9], page 95 for more details).

**Theorem 2.4.** Let $S$ be a semigroup with an identity and $T \in \text{Mul}_l(S)$. If $S_T$ is semisimple, then $S$ is so. The converse is true if $S_T$ is left cancellative and $T$ is surjective.

**Proof.** Since $S$ is unital there exists $\mu \in S$ such that $T = L_\mu$. Suppose that $S_T$ is semisimple and $I$ is an ideal of $S$. Then

$$I \circ S = I \cap (S(T) \subseteq I S \subseteq I).$$

Similarly, $S \circ I \subseteq I$. It follows that $I$ is an ideal of $S_T$. By the hypothesis $(I_T)^2 = I \circ I = I$. Now, take $r \in I$ then there are $s, t \in I$ such that

$$r = s \circ t = sT(t) = s(\mu t) \in I^2.$$

So we show that $I^2 = I$ and hence $S$ is semisimple.

Conversely, assume that $S_T$ is left cancellative and $T \in \text{Mul}_l(S)$ is surjective then by theorem 2.1(ii), $T^{-1} \in \text{Mul}_l(S)$. So, there exists $b \in S$ such that $T^{-1} = L_b$. Suppose that $\bar{S} = S_{T^{-1}}$. Then we have

$$S = S_{T^{-1}} = (S_T)_{T^{-1}} = \bar{S}_{T^{-1}}.$$

By hypothesis and above the proof, $\bar{S} = S_{T^{-1}}$ is semisimple.

**Semigroup Algebra $\ell^1(S_T)$**

We say that a discrete semigroup $S$ is amenable if there exists a positive linear functional on $\ell^\infty(S)$ called a mean such that

$$m(1) = 1 = m(l_s f) = m(f), m(r_s f) = m(f)$$

for each $s \in S$, where $l_s f(t) = f(st)$ and $r_s f(t) = f(ts)$ for all $t \in S$. The definition of amenable group is similar to semigroup case. Refer to [12] for more details.
Let \( \mathcal{A} \) be a Banach algebra and let \( X \) be a Banach \( \mathcal{A} \)–bimodule. A derivation from \( \mathcal{A} \) to \( X \) is a linear map \( D: \mathcal{A} \to X \) such that
\[
D(ab) = D(a) \cdot b + a \cdot D(b) \quad (a, b \in \mathcal{A}).
\]
A derivation \( D \) is inner if there exists \( x \in X \) such that
\[
D(a) = a \cdot x - x \cdot a \quad (a \in \mathcal{A}).
\]
The Banach algebra \( \mathcal{A} \) is amenable if every bounded derivation \( D: \mathcal{A} \to X^* \) is inner for all Banach \( \mathcal{A} \)–bimodule \( X \). Where \( X^* \) is the dual space of \( X \). We say that the Banach algebra \( \mathcal{A} \) is weakly amenable if any bounded derivation \( D \) from \( \mathcal{A} \) to \( \mathcal{A}^* \) is inner. For more details see [12], [16].

If \( S \) is a commutative semigroup, by theorem 5.8 of [8] \( \ell^1(S) \) is called semisimple if and only if for all \( x, y \in S \), \( x^2 = y^2 = xy \) implies \( x = y \).

**Theorem 3.1.** Let \( S \) be a commutative semigroup and let \( T \in \text{Mul}_l(S) \) be injective. Then \( \ell^1(S) \) is semisimple if and only if \( \ell^1(S_T) \) is semisimple.

**Proof.** Take \( r, s \in S \). Then \( r^2 = s^2 = rs \) if and only if \( T(r^2) = T(s^2) = T(r)T(s) \) or equivalently \( r_0 r = s_0 s = r_0 s \), because \( T \) is injective. So, by theorem 5.8 [8], \( \ell^1(S) \) is semisimple if and only if \( \ell^1(S_T) \) is semisimple.

**Theorem 3.2.** Let \( S \) be a discrete semigroup and \( T \in \text{Mul}_l(S) \). Then
(i) The left multiplier \( T \) has an extension \( \bar{T} \in \text{Mul}_l \left( \ell^1(S) \right) \) with the norm decreasing.
(ii) The left multiplier \( T \) is injective if and only if so is \( \bar{T} \).
(iii) If \( T \) is injective then \( \bar{T} \) is an isometry and also \( \ell^1(S_T) \) and \( \left( \ell^1(S) \right)_T \) are isomorphic.

**Proof.** (i) An arbitrary element \( f \in \ell^1(S) \) is of the form \( f: S \to \mathbb{C} \) such that \( f(x) = 0 \) except at the most countable subset \( A \) of \( S \). If \( A \) is a finite subset of \( S \) then \( f = \sum_{k=1}^{n} f(x_k) \delta_{x_k} \) for some fixed \( n \in \mathbb{N} \). So in general we have
\[
f = \sum_{x \in S} f(x) \delta_x = \sum_{x \in A} f(x) \delta_x = \sum_{k=1}^{\infty} f(x_k) \delta_{x_k}.
\]
Now, for each \( n \in \mathbb{N} \), let \( f_n = \sum_{k=1}^{n} f(x_k) \delta_{x_k} \) and define \( \bar{T}: \ell^1(S) \to \ell^1(S) \) by
\[
\bar{T}(\delta_x) = \delta_{T(x)} \quad (x \in S),
\]
\[
\bar{T}(f_n) = \sum_{k=1}^{n} f(x_k) \delta_{x_k} = \bar{f}_n.
\]
For each \( m, n \in \mathbb{N} \) where \( n \geq m \), we have
\[
\| \bar{T}(f_n) - \bar{T}(f_m) \|_1 = \| \bar{f}_n - \bar{f}_m \|_1 = \| \sum_{k=m}^{n} f(x_k) \delta_{x_k} \|_1 = \| \sum_{k=m}^{n} f(x_k) \delta_{T(x_k)} \|
\]
\[
\leq \sum_{k=m}^{n} |f(x_k)| = \| f_n - f_m \|_1.
\]
So \( \{\tilde{T}(f_n)\}_n \) is a Cauchy sequence and it is convergent. Now, we define \( \tilde{T}(f) = \lim_n \tilde{T}_n \).

Then the definition is well defined. Hence
\[
\tilde{T}(f) = \sum_{k=1}^{\infty} f(x_k) \delta_{x_k} = \tilde{f}.
\]
also
\[
\|\tilde{f}\|_1 = \sum_{x_k \in A} |f(x_k)| = \|f\|_1 \quad \text{or} \quad \|\tilde{T}(f)\|_1 \leq \|f\|_1.
\]
It shows that \( \tilde{T} \) is norm decreasing.

In the following, we extend \( \tilde{T} \) by linearity. Let \( f, g \in \ell^1(S) \). Then there are two at most countable sub set \( A, B \) of \( S \) such that
\[
f = \sum_{x \in A} f(x) \delta_x, \quad g = \sum_{x \in B} g(x) \delta_x.
\]
Suppose that \( D = A \cup B \). So we have \( f + g = \sum_{x \in D} (f(x) + g(x)) \delta_x \).

Then, it follows that
\[
\tilde{T}(f + g) = \tilde{f} + \tilde{g} = \sum_{x \in D} (f(x) + g(x)) \tilde{T}(\delta_x) = \sum_{x \in A} f(x) \tilde{T}(\delta_x) + \sum_{x \in B} g(x) \tilde{T}(\delta_x)
\]
\[
= \tilde{f} + \tilde{g}.
\]
Also, if \( \alpha \in \mathbb{C} \), we have
\[
\tilde{T}(\alpha f) = \alpha \tilde{T}(f) = \sum_{x \in A} \alpha f(x) \tilde{T}(\delta_x) = \alpha \sum_{x \in A} f(s) \tilde{T}(\delta_x) = \alpha \tilde{T}(f).
\]
Therefore, \( \tilde{T} \) is a bounded linear isometry.

Now, we prove that \( \mathcal{T} \in \text{Mul}_1(\ell^1(S)) \). Take \( x, y \in S \). Then
\[
\tilde{T}(\delta_x * \delta_y) = \tilde{T}(\delta_{xy}) = \delta_{T(xy)} = \delta_{T(x)y} = \delta_{T(x)} * \delta_y = \tilde{T}(\delta_x) * \delta_y.
\]
Let \( y \in S \) be fixed and \( f, g \in \ell^1(S) \). Then
\[
\tilde{T}(f * \delta_y) = \tilde{T} \left( \sum_{x \in A} f(x) \delta_{xy} \right) = \sum_{x \in A} f(x) \tilde{T}(\delta_{xy})
\]
\[
= \left( \sum_{x \in A} \tilde{T}(\delta_x) \right) * \delta_y = \tilde{T}(f) * \delta_y.
\]
In the general case, we have
\[
\tilde{T}(f * g) = \tilde{T} \left( \sum_{x \in A} f(x) (\sum_{y \in B} g(y)) \delta_{xy} \right)
\]
\[
= \sum_{x \in A} f(x) \tilde{T}(\delta_x) * \sum_{y \in B} g(y) \delta_y = \tilde{T}(f) * g.
\]
This shows that \( \tilde{T} \) is a multiplier on \( \ell^1(S) \).

(ii) Let \( T \) be injective. Take \( x, y \in S \) and suppose that \( \tilde{T}(\delta_x) = \tilde{T}(\delta_y) \). Then \( \delta_{T(x)} = \tilde{T}(\delta_x) = \tilde{T}(\delta_y) = \delta_{T(y)} \).

Therefore, \( T(x) = T(y) \). Since \( T \) is injective, we have \( x = y \). It follows that \( \delta_x = \delta_y \), consequently \( \tilde{T} \) is injective.

Conversely, the same argument shows that the converse holds.

(iii) Let \( T \) be injective and \( f \in \ell^1(S) \). Then there exists at most a countable subset \( A \subseteq S \) such that
Since $A$ and $T(A)$ have the same cardinal number, $\|\tilde{T}(f)\| = \|\sum_{x \in A} f(x) \delta_x\| = \sum_{x \in A}|f(x)| = \|f\|_1$, so $\tilde{T}$ is an isometry.

Now, we can define a new multiplication "\(\boxplus\)" on $\ell^1(S)$ as follow
\[
f \boxplus g = f \ast \tilde{T}g \quad (f, g \in \ell^1(S)).
\]
By a similar argument in theorem 1.31 [10], $\ell^1(S)$ with the new product is a Banach algebra that is denoted it by $\ell^1(S)_{\tilde{T}}$. We define the map $\Psi: \ell^1(S_T) \to \ell^1(S)_{\tilde{T}}$, by $\Psi(\delta_x) = \delta_x \quad (x \in S)$.

Take $x, y \in S$. Then
\[
\Psi(\delta_x \ast \delta_y) = \Psi(\delta_{x \ast y}) = \delta_{x_{T(y)}} = \delta_x \ast \delta_{T(y)}
\]
\[
= \delta_x \ast \tilde{T}(\delta_y) = \delta_x \boxplus \delta_y
\]
\[
= \Psi(\delta_x) \boxplus \Psi(\delta_y).
\]

So, in general case, we have
\[
\Psi(f \ast g) = \Psi(f) \boxplus \Psi(g) \quad (f, g \in \ell^1(S)).
\]
Thus, $\Psi$ is an isomorphism. Therefore $\ell^1(S_T)$ and $\ell^1(S)_{\tilde{T}}$ are isomorphic.

**Theorem 3.3.** Let $S$ be a semigroup and $T \in Mul_f(S)$ be bijective. Then $\ell^1(S)$ is amenable if and only if $\ell^1(S_T)$ is amenable.

**Proof.** By theorem 3.2, we have $\ell^1(S_T) \cong \ell^1(S)_{\tilde{T}}$. Suppose that $\ell^1(S_T)$ is amenable and define $\varphi: \ell^1(S_T) \to \ell^1(S)$ by $\varphi(f) = \tilde{T}(f)$. Take $x, y \in S$. Then
\[
\varphi(\delta_x \boxplus \delta_y) = \tilde{T}(\delta_x \boxplus \delta_y) = \tilde{T}(\delta_{x_{T(y)}}) = \tilde{T}(\delta_x \ast \delta_{T(y)}) = \tilde{T}(\delta_x) \ast \delta_{T(y)}
\]
\[
= \tilde{T}(\delta_x) \ast \tilde{T}(\delta_y) = \varphi(\delta_x) \ast \varphi(\delta_y).
\]
Now, by induction and continuity of $\tilde{T}$, we have
\[
\varphi(f \boxplus g) = \varphi(f) \ast \varphi(g).
\]
If $T$ is bijective, $\tilde{T}$ is bijective. Therefore $\varphi$ is an epimorphism of $\ell^1(S_T)$ onto $\ell^1(S)$.

Hence, by proposition 2.3.1 [16] $\ell^1(S)$ is amenable.

Conversely, suppose that $\ell^1(S)$ is amenable. Since $T$ is bijective, $\tilde{T}$ is bijective. Therefore $\tilde{T}^{-1}$ exists. Now define $\theta: \ell^1(S) \to \ell^1(S_T) \cong \ell^1(S)_{\tilde{T}}$ by $\theta(f) = \tilde{T}^{-1}(f)$.

Take $x, y \in S$. Then
\[
\theta(\delta_x \ast \delta_y) = \tilde{T}^{-1}(\delta_{xy}) = \tilde{T}^{-1}(\delta_x)\tilde{T}^{-1}(\delta_y) = \tilde{T}^{-1}(\delta_x) \boxplus \tilde{T}^{-1}(\delta_y)
\]
\[
= \theta(\delta_x) \boxplus \theta(\delta_y).
\]
Similarly \( \theta \) is an epimorphism from \( \ell^1(S) \) onto \( \ell^1(S_T) \). By proposition 2.3.1 [16] \( \ell^1(S_T) \) is amenable.

Note that, in general, it is not known when \( \ell^1(S) \) is weakly amenable. For more details see [2].

**Theorem 3.4.** Let \( S \) be a semigroup and \( T \in \text{Mul}_l(S) \) be bijective. Then, if \( S \) is completely regular then \( \ell^1(S_T) \) is weakly amenable.

**Proof.** It is enough to prove that \( S_T \) is completely regular, then by theorem 3.6 [2], \( \ell^1(S_T) \) can be weakly amenable. Take \( s \in S \). Then there exists \( r \in S \) such that \( T(s) = T(s)T(r)T(s) \), \( T(r)T(s) = T(s)T(r) \), since \( T \) is bijective and \( S = T(S) \) is completely regular. So we have \( T(s) = T(s \circ r \circ s) \) and \( T(r \circ s) = T(s \circ r) \). Hence \( s = s \circ r \circ s \) and \( r \circ s = s \circ r \) for some \( r \in S \), since \( T \) is injective. Therefore \( S_T \) is completely regular.

**Corollary 3.5.** Suppose that \( S \) is a commutative completely regular semigroup and \( T \in \text{Mul}_l(S) \) is injective. Then \( \ell^1(T(S)_T) \) is weakly amenable.

**Proof.** [2, theorem 3.6] \( \ell^1(S) \) is weakly amenable. Define \( \varphi : S \to \ell^1(S)_T \) by

\[
\varphi(s) = T^{-1}(s) \quad (s \in S).
\]

We show that \( \varphi \) is a homomorphism. Take \( s \in S \), then we have

\[
\varphi(rs) = T^{-1}(rs) = T^{-1}(r) s = T^{-1}(r) \circ (T^{-1}s).
\]

So \( \varphi \) is a homomorphism. Then by proposition 2.1[7], \( \ell^1(T(S)_T) \) is weakly amenable.

In the case that \( S \) is a group, it is easy to see that the amenability of \( S \) implies the amenability of \( \ell^1(S_T) \). Indeed, when \( S \) is a group, by theorem 2.1, \( S_T \) is a semigroup and one can easily prove that \( S_T \) is also a group. On the other hand, \( \text{Mul}_l(S) \cong S \) because \( S \) is a unital semigroup, so each \( T \in \text{Mul}_l(S) \) is inner and of the form \( T = L_s \) for some \( s \in S \). Also \( T^{-1} = L_{a^{-1}} \) exists, since \( S \) is a group. Then the map \( \theta : S_T \to S \) defined by \( \theta(s) = T(s) \) is an isomorphism; that is \( S \cong S_T \). Thus we have the following result:

**Corollary 3.6.** Let \( S \) be a cancellative regular discrete semigroup. Then \( \ell^1(S) \) is amenable if and only if \( \ell^1(S_T) \) is amenable.

**Proof.** By [9, Exercise 2.6.11] \( S \) is a group. So the assertion holds by [15, theorem 2.1.8]
Examples

In this section we present some examples which either comments on our results or indicate necessary condition in our theorems.

4.1. There are semigroups $S$ and $T \in \text{Mul}_l(S)$ such that the background semigroups $S$ are not commutative but their induced semigroups $S_T$ are commutative.

This example shows that the condition $\overline{T(S)} = S$, in theorem 2.2, can not be omitted.

Let $S$ be the set $\{a, b, c, d, e\}$ with operation table given by

<table>
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<tr>
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<th>a</th>
<th>b</th>
<th>c</th>
<th>d</th>
<th>e</th>
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</table>

Clearly $(S, \cdot)$ is a non-commutative semigroup. Now, put $T = L_a$ where $L_a(x) = ax$ for all $x \in S$. One can get easily the operation table of $S_T$ as follow:

<table>
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</tbody>
</table>

The operation table shows that the induced semigroup $S_T$ is commutative and $T(S) \neq S$. Also the other induced semigroup $S_T$ is commutative for $T = L_d$ analogously.

Now we present some important theorems from[14] that we need in the following examples:

**Theorem 4.2.** Let $S$ be a semigroup. Suppose that $\ell^1(S)$ is amenable. Then

(i) $S$ is amenable
(ii) $S$ is regular.
(iii) $E(S)$ is finite.
(iv) $\ell^1(S)$ has an identity.

**Proof.** (i) That is lemma 3 in [5].
(ii) and (iii) See theorem 2 in[6].
(iv) That is corollary 10.6 in [4].

**Theorem 4.3.** Let $S$ be a finite semigroup. Then the following statements are equivalent:

(i) $\ell^1(S)$ is amenable.

(ii) $S$ is regular and \( \ell^1(S) \) is nuietal.

(ii) $S$ is regular and \( \ell^1(S) \) is semisimple.

**Proof.** Refer to [3].

4.4. There are semigroups $S$ and $T \in \operatorname{Mul}(S)$ such that $S$ and $\ell^1(S)$ are amenable but $S_T$ is not regular and also, $\ell^1(S_T)$ is not amenable.

This example shows that two semigroup algebras $\ell^1(S)$ and $\ell^1(S_T)$ can be different in some properties. Also, it notifies that the bijectivity of $T$ in the theorem 3.3 is essential. Put $S = \{x_0, x_1, x_2, \ldots, x_n\}$ with the operation $x_i x_j = x_{\max\{i, j\}}$ $(0 \leq i, j \leq n, n \geq 2)$.

Then $S$ is a semigroup. Since

$$\operatorname{Max}\{i, \operatorname{Max}\{j, k\}\} = \operatorname{Max}\{\operatorname{Max}\{i, j\}, k\} = \operatorname{Max}\{i, j, k\}.$$ 

We denote it by $S_v$. This semigroup is commutative. So by (0.18) in [12], it is amenable. $S_v$ is a unital semigroup and has a zero; indeed, $e_s = x_0$ and $a_s = x_n$. Also, it is a regular semigroup and $\operatorname{Mul}(S_v) \cong S_v$ because $S_v$ has an identity.

Evidently, $S_v$ is regular since each $s \in S_v$ is idempotent. The semigroup algebra $\ell^1(S_v)$ is a unital algebra because $S_v$ has an identity. So by theorem 4.3 (ii) $\ell^1(S_v)$ is amenable.

Now, take $T = L_{x_k}$ for a fixed $x_k \in S$ where $k \geq 1$. By theorem 2.2, $(S_v)_T$ is commutative so is amenable. We show that $T$ is neither injective and surjective.

Take $x_i \in S_v$, then $Tx_i = x_k x_i = x_{\max\{k, i\}}$. So

$$T(S_v) = \{x_k, x_{k+1}, \ldots, x_n\} \neq S_v.$$ 

Hence, $T$ is not surjective.

Again, take distinct elements $x_i, x_j$ in $S_v$ for some $i, j < k$ such that $T(x_i) = T(x_j).$ Then we have $x_{\max\{k, i\}} = x_{\max\{k, j\}}$ but $x_i \neq x_j.$ So $T$ is not injective.

We prove that $(S_v)_T$ is not regular. If $(S_v)_T$ is regular, then for $x_{k-1} \in S_v$ there exists an element $x_j \in S_v$ such that

$$x_{k-1} = x_{k-1} o x_j o x_{k-1} = x_{\max\{k, j\}}.$$
That implies that $\max\{k,j\} = k - 1$; which is impossible. Consequently, by theorem 4.2 (ii) or 4.3 (ii), $\ell^1((S_v)_T)$ is not amenable.

Also, the inequality $S_v \circ S_v = \{x_k, x_{k+1}, ..., x_n\} \neq S_v$ shows that $\ell^1((S_v)_T)$ is not weakly amenable. In the next example we show that in the theorem 3.2 (iii) the condition "injectivity of " can not be omitted.

4.5 There are a semigroup $S$ and $T \in Mul_1(S)$ such that $T \in Mul_1(S)$ is not injective and the corresponding $\overline{T} \in Mul_1\left(\ell^1(S_T)\right)$ is not an isometry.

Suppose that $S_v$ is a semigroup as in example 4.4 and $T = L_{x_k}$ for some fixed $1 < k < n$. If $f \in \ell^1(S_v)$ then $f = \sum_{i=0}^{n} f(x_i)\delta_{x_i}$ and also $\overline{T}(f) = \sum_{i=0}^{n} f(x_i)\delta_{\overline{T}(x_i)}$. But $T(x_i) = \{x_i \mid k < i \leq n, x_k \mid 0 \leq i \leq k\}$, so

$$T(f) = \left(\sum_{i=0}^{k} f(x_i)\right)\delta_{x_k} + \sum_{i=k+1}^{n} f(x_i)\delta_{\overline{T}(x_i)}.$$

Hence

$$\|T(f)\| = \left|\sum_{i=0}^{k} f(x_i)\right| + \sum_{i=k+1}^{n} |f(x_i)| \leq \sum_{i=0}^{k} |f(x_i)| + \sum_{i=k+1}^{n} |f(x_i)| = \|f\|_1.$$

It shows that $\overline{T}$ is not an isometry.

4.6. There are semigroups $S$ and $T \in Mul_1(S)$ such that $\ell^1(S)$ is semisimple. But $\ell^1(S_T)$ is not semisimple. This example remind that, in theorem 3.1 the multiplier $T$ must be injective.

Let $S$ be a set $\{x_0, x_1, ..., x_n\}$ where $n \in \mathbb{N}$ and $n \geq 3$ is fixed. By operation given by $xy = x_{\min\{i,j\}}$, $S$ is a commutative semigroup. Since

$$\min\{i,\min\{j,k\}\} = \min\{\min\{i,j\},k\} = \min\{i,j,k\} \quad (i,j,k \in \mathbb{N}).$$

We denote it briefly by $S_\wedge$. For each $x,y \in S$ the equality $x^2 = y^2 = xy$ implies $x=y$. So by Theorem 5.8 $\ell^1(S_\wedge)$ is semisimple.

Now, let $T = L_{x_k}$ for a fixed $1 \leq k < n - 1$. It is easy to see that $T(x_k) = T(x_n)$ but $x_k \neq x_n$. So the multiplier $T$ is not injective.

We show that neither $S_\wedge$ nor $\ell^1(S_\wedge)_T$ is semisimple.

Each ideal of $S$ is of the form

$$I_m = \{x_0, x_1, ..., x_m\} \quad (m \leq n).$$

We claim that $S_T$ is not semisimple. Since for each $m \in \mathbb{N}$ we have
Amenability and Weak Amenability of The Semigroup Algebra \( \ell^1(S_a) \)

\[
I_m \circ I_m = \begin{cases} 
I_m & m \leq k \\
I_k & m > k 
\end{cases}
\]

On the other hand, for each \( x_i, x_j \in S \) where \( i \neq j \) and \( i, j > k \), we have \( x_i \circ x_i = x_j \circ x_j = x_k \), while \( x_i \neq x_j \). Thus, Theorem 5.8 [8] shows that \( \ell^1(S_a)_T \) is not semisimple.

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**Reference**


