Amenability and Weak Amenability of the Semigroup Algebra $\ell^1(S_T)$

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Abstract

Let $S$ be a semigroup with a left multiplier $T$ on $S$. A new product on $S$ is defined by $T$ related to $S$ and $T$ such that $S$ and the new semigroup $S_T$ have the same underlying set as $S$.

It is shown that if $T$ is injective then $\ell^1(S_T) \equiv \ell^1(S)_{T\bar{}}$ where, $\bar{T}$ is the extension of $T$ on $\ell^1(S)$. Also, we show that if $T$ is bijective, then $\ell^1(S)$ is amenable if and only if $\ell^1(S_T)$ is so. Moreover, if $S$ completely regular, then $\ell^1(S_T)$ is weakly amenable.

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Introduction

Let $S$ be a semigroup and $T$ be a left multiplier on $S$. We present a general method of defining a new product on $S$ which makes $S$ a semigroup. Let $S_T$ denote $S$ with the new product. These two semigroups are sometimes different and we try to find conditions on $S$ and $T$ such that the semigroups $S$ and $S_T$ have the same properties. This idea has started by Birtel in [1] for Banach algebras and continued by Larsen in [11]. Recently, this notion developed by some authors, for more details see [1], [10], [11], [12] and [15]. One of the best result in this work expresses that $L^1(G)_T$ is Arens regular if and only if $G$ is a compact group [10]. We continue this direction on the regularity of $S$ and $S_T$ and the amenability of their semigroup algebras.

The term of semigroup will be a non-empty set $S$ endowed with an associative binary operation on $S$, defined by $(s,t) \rightarrow st$. If $S$ is also a Hausdorff topological space and the binary operation is jointly continuous, then $S$ is called a topological semigroup.

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Let $p \in S$. Then $p$ is an idempotent if $p^2 = p$. The set of all idempotents of $S$ is denoted by $E(S)$.

An element $e$ is a left (right) identity if $es = s$ (resp. $se = s$) for all $s \in S$. An element $ecS$ is an identity if it is a left and a right identity. An element $z$ is a left (resp. right) zero if $zs = z$ (resp. $sz = z$) for all $s \in S$. An element $zcS$ is a zero if it is a left and a right zero. We denote any zero of $S$ by $0_S$ (or $z_S$). An element $pcS$ is a regular element of $S$ if there exists $tcS$ such that $p = ptp$ and $p$ is completely regular if it is regular and $pt = tp$. We say that $p \in S$ has an inverse if there exists $tcS$ such that $p = ptp$ and $t = tpt$. Note that the inverse of element $p \in S$ need not be unique. If $pcS$ has an inverse, then $p$ is regular and vice versa. Since, if $p \in S$ is regular, there exists $s \in S$ such that $p = psp$. Let $t = sps$. Then

$$p = psp = (p(s)p)s = p = ptp, \quad t = sps = (s(p)s)p = (sp)s(p(s)) = tpt.$$ 

So $p$ has an inverse. We say that $S$ is a regular (resp. completely regular) semigroup if each $pcS$ is regular (resp. completely regular). Also $S$ is an inverse semigroup if each $p \in S$ has a unique inverse. The map $T : S \rightarrow S$ is called a left (resp. right) multiplier if

$$T(st) = T(s)t \quad (\text{resp. } T(st) = sT(t)) \quad (s, tcS).$$ 

The map $T : S \rightarrow S$ is a multiplier if it is a left and right multiplier. Let $S$ be a topological semigroup. The net $(e_\alpha) \subseteq S$ is a left (resp. right) approximate identity if $\lim_\alpha e_\alpha t = t$. (resp. $\lim_\alpha t e_\alpha = t$) (tcS). The net $(e_\alpha) \subseteq S$ is an approximate identity if it is a left and a right approximate identity.

Let $S$ be a discrete semigroup. We denote by $\ell^1(S)$ the Banach space of all complex function $f : S \rightarrow \mathbb{C}$ having the form

$$f = \sum_{s \in S} f(s) \delta_s,$$

such that $\sum_{s \in S}|f(s)| = \|f\|_1$ is finite, where $\delta_s$ is the point mass at $\{s\}$. For $f, g \in \ell^1(S)$ we define the convolution product on $\ell^1(S)$ as follow:

$$f * g(s) = \sum_{t_1 + t_2 = s} f(t_1)g(t_2) \quad (scS),$$

with this product $\ell^1(S)$ becomes a Banach algebra and is called the semigroup algebra on $S$.

Remark 1.1. If $f \in \ell^1(S)$ then $f = 0$ on $S$ except at most on a countable subset of $S$. In other words, the set $A = \{scS : f(s) \neq 0\}$ is at most countable. Since, if $A_n = \{scS : |f(s)| \geq \frac{1}{n}\}$, $A = \bigcup_{n \in \mathbb{N}} A_n$. Set $\|f\|_1 = M$ and $n \in \mathbb{N}$ is fixed. Then we have

$$M = \sum_{s \in S} |f(s)| \geq \sum_{s \in A_n} |f(s)| \geq \sum_{s \in A_n} \frac{1}{n} = \frac{1}{n} |A_n|,$$
where \(|A_n|\) is the cardinality of \(A_n\). So \(|A_n| \leq nM\). Hence \(A_n\) is a finite subset of \(S\) and thus \(A\) is at most countable.

**Semigroup \(S_T\)**

Let \(T \in \text{Mul}_1(S)\). Then we define a new binary operation "\(\circ\)" on \(S\) as follow:

\[
s \circ t = s \cdot T(t) \quad (s, t \in S).
\]

The set \(S\) equipped with the new operation "\(\circ\)" is denoted by \(S_T\) and sometimes called "induced semigroup of \(S\)". Now we have the following results.

**Theorem 2.1.** Let \(S\) be a Semigroup. Then (i) if \(T \in \text{Mul}_1(S)\) then \(S_T\) is a semigroup.

The converse is true if \(S\) is left cancellative and \(T\) is surjective.

(ii) If \(S_T\) is left cancellative and \(T\) is surjective, then \(T^{-1} \in \text{Mul}_1(S)\).

(iii) If \(S\) is a topological semigroup and \(S_T\) has a left approximate identity then \(T^{-1} \in \text{Mul}_1(S)\).

**Proof.**

i) Let \(T \in \text{Mul}_1(S)\) and take \(r, s, t \in S\). Then

\[
(r \circ (s \circ t)) = r \cdot T(s \circ t) = r \cdot T(s \cdot T(t)) = r \cdot T(s) \cdot T(t) = (r \cdot T(s)) \cdot T(t) = (r \circ s) \circ t.
\]

So, \(S_T\) is a semigroup.

Conversely, suppose that \(S\) is left cancellative and take \(r, s, t \in S\). Since \(T\) is surjective, there exists \(u \in S\) such that \(T(u) = t\). Then

\[
r \cdot T(st) = r \cdot T(sT(u)) = r \circ (s \circ u) = (r \circ s) \circ u = (rT(s)) \cdot T(u).
\]

By the left cancellativity of \(S\), we have \(T(st) = T(s)T(t) = (r, s \circ S)\). So, \(T\) is a left multiplier.

ii) We must prove that \(T\) is injective. To do this end, take \(r, s, u \in S\) and let \(T(r) = T(s)\).

Then \(u \circ r = uT(r) = uT(s) = u \circ s\). So \(r = s\), since \(S_T\) is left cancellative. Hence \(T^{-1}\) exists.

Now, we show that \(T^{-1} \in \text{Mul}_1(S)\). Take \(r, s \in S\). Then

\[
T^{-1}(rs) = T^{-1}[TT^{-1}(r)s] = T^{-1}[T(T^{-1}(r)s)] = (T^{-1}T)[T^{-1}(r)s] = T^{-1}(r)s.
\]

iii) It is enough to show that \(T\) is injective. Take \(r, s \in S\) and suppose that \(T(r) = T(s)\). Then

\[
r = \lim_\alpha e_\alpha \circ r = \lim_\alpha e_\alpha T(r) = \lim_\alpha e_\alpha T(s) = \lim_\alpha e_\alpha \circ s = s.
\]
There are many properties that induced from $S$ to semigroup $S_T$. But sometimes they are different.

**Theorem 2.2.** Let $S$ be a Hausdorff topological semigroup and $\text{TeMul}_1(S)$. If $S$ is commutative then so is $S_T$. The converse is true if $\overline{T(S)} = S$.

**Proof.** Suppose $S$ is commutative and take $r, s \in S$. Then
\[
rs = rT(s) = T(s)r = T(sr) = T(rs) = sT(r) = s\circ r.
\]
So, $S_T$ is commutative.

Conversely, Let $S_T$ be commutative and take $r, s \in S$. Then there exist nets $(r_\alpha)$ and $(s_\beta)$ in $S$ such that $\lim_\alpha T(r_\alpha) = r$ and $\lim_\beta T(s_\beta) = s$.

So, we have
\[
rs = \lim_\alpha \lim_\beta T(r_\alpha \circ s_\beta) = \lim_\alpha \lim_\beta T(s_\beta \circ r_\alpha) = \lim_\alpha \lim_\beta T(s_\beta) T(r_\alpha) = s\circ r.
\]
Thus $S$ is commutative.

In the sequel, we investigate some relations between two semigroup $S$ and $S_T$ according to the role of the left multiplier $T$.

**Theorem 2.3.** Let $S$ be a semigroup and $\text{TeMul}_1(S)$. Then

(i) If $T$ is surjective and $S_T$ is an inverse semigroup then $S$ is an inverse semigroup and $T(s^{-1}) = T(s)^{-1}$ for all $s \in S$.

(ii) If $S_T$ is an inverse semigroup and $T$ is injective then $T(S)$ is an inverse subsemigroup of $S$.

(iii) If $T$ is bijective then $S_T$ is an inverse semigroup if and only if $S$ is an inverse semigroup.

**Proof.** i) Suppose that $S_T$ is an inverse semigroup and $T$ is surjective. Define the map $\varphi: S_T \to S$ by $\varphi(s) = T(s)$. Take $r, s \in S$, then
\[
\varphi(r\circ s) = T(r\circ s) = T(r)T(s) = \varphi(r)\varphi(s).
\]
So, $\varphi$ is an epimorphism from $S_T$ onto $S$, since $T$ is surjective. By theorem 5.1.4[7], $S$ is an inverse semigroup and $T(s^{-1}) = T(s)^{-1}$ for all $s \in S$.

ii) Suppose that $T$ is injective and $S_T$ is an inverse semigroup. Evidently, $T(S)$ is a subsemigroup of $S$. We show that it is an inverse semigroup. Take $s \in T(S)$. There exists $t \in S$ such that $s = T(t)$. Also, there exists a unique element $u \in S$ such that $t = t_0 u_0 t$, since $S_T$ is an inverse semigroup. Therefore, $T(t) = T(t)T(u)T(t)$ or $s = s_0 T(u)_0 s$. Of course, $T(u)$ is unique because $u \in S$ is unique and $T$ is injective. Hence $T(S)$ is an inverse subsemigroup of $S$. 

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iii) Suppose that $T$ is bijective and let $S_T$ be an inverse semigroup. Since $T$ is injective and surjective, by (i) and (ii), $S = T(S)$ is an inverse semigroup.

Conversely, suppose that $S$ is an inverse semigroup. Since $T$ is bijective, by theorem 2.1(ii), $T^{-1} \in \text{Mul}_l(S)$. So $\varphi^{-1}: S \rightarrow S_T$ defined by $\varphi^{-1}(s) = T^{-1}(s)$ is an epimorphism. Hence by (i) $S_T$ is an inverse semigroup.

We say that $T \in \text{Mul}_l(S)$ is an inner left multiplier if it has the form $T = L_s$ for some $s \in S$ where $L_s(t) = st$ $(teS)$.

If $T \in \text{Mul}_l(S)$ is inner, then each ideal of $S$ is permanent under $T$; that is $T(I) \subseteq I$ for all ideal $I$ of $S$. It is easily to see that if $S$ has an identity, then each $T \in \text{Mul}_l(S)$ is inner.

Let $S$ be a semigroup. Then $S$ is called semisimple if $I^2 = I$ for all ideal $I$ of $S$ (see [9], page 95 for more details).

**Theorem 2.4.** Let $S$ be a semigroup with an identity and $T \in \text{Mul}_l(S)$. If $S_T$ is semisimple, then $S$ is so. The converse is true if $S_T$ is left cancellative and $T$ is surjective.

**Proof.** Since $S$ is unital there exists $\mu \in S$ such that $T = L_\mu$. Suppose that $S_T$ is semisimple and $I$ is an ideal of $S$. Then

$$I \circ S = I \circ T(S) \subseteq I \circ S \subseteq I.$$ 

Similarly, $S \circ I \subseteq I$. It follows that $I$ is an ideal of $S_T$. By the hypothesis $(I_T)^2 = I \circ I = I$. Now, take $r \in I$ then there are $s, t \in I$ such that

$$r = s \circ t = sT(t) = s(\mu t) \in I^2.$$ 

So we show that $I^2 = I$ and hence $S$ is semisimple.

Conversely, assume that $S_T$ is left cancellative and $T \in \text{Mul}_l(S)$ is surjective then by theorem 2.1(ii), $T^{-1} \in \text{Mul}_l(S)$. So, there exists $b \in S$ such that $T^{-1} = L_b$. Suppose that $\tilde{S} = S_T^{-1}$. Then we have

$$S = S_{T^{-1}} = (S_T)_T^{-1} = \tilde{S}_{T^{-1}}.$$ 

By hypothesis and above the proof, $\tilde{S} = S_{T^{-1}}$ is semisimple.

**Semigroup Algebra** $\ell^1(S_T)$

We say that a discrete semigroup $S$ is amenable if there exists a positive linear functional on $\ell^\infty(S)$ called a mean such that $m(1) = 1$ and $m(l_s f) = m(f), m(r_s f) = m(f)$ for each $s \in S$, where $l_s f(t) = f(st)$ and $r_s f(t) = f(ts)$ for all $teS$. The definition of amenable group is similar to semigroup case. Refer to [12] for more details.
Let $\mathfrak{A}$ be a Banach algebra and let $X$ be a Banach $\mathfrak{A}$–bimodule. A derivation from $\mathfrak{A}$ to $X$ is a linear map $D: \mathfrak{A} \to X$ such that
\[
D(ab) = D(a) \cdot b + a \cdot D(b) \quad (a, b \in \mathfrak{A}).
\]
A derivation $D$ is inner if there exists $x \in X$ such that
\[
D(a) = a \cdot x - x \cdot a \quad (a \in \mathfrak{A}).
\]

The Banach algebra $\mathfrak{A}$ is amenable if every bounded derivation $D: \mathfrak{A} \to X^*$ is inner for all Banach $\mathfrak{A}$–bimodule $X$. Where $X^*$ is the dual space of $X$. We say that the Banach algebra $\mathfrak{A}$ is weakly amenable if any bounded derivation $D$ from $\mathfrak{A}$ to $\mathfrak{A}^*$ is inner. For more details see [12], [16].

If $S$ is a commutative semigroup, by theorem 5.8 of [8] $\ell^1(S)$ is called semisimple if and only if for all $x, y \in S$, $x^2 = y^2 = xy$ implies $x = y$.

**Theorem 3.1.** Let $S$ be a commutative semigroup and let $T \in Mul_l(S)$ be injective. Then $\ell^1(S)$ is semisimple if and only if $\ell^1(S_T)$ is semisimple.

**Proof.** Take $r, s \in S$. Then $r^2 = s^2 = rs$ if and only if $T(r^2) = T(s^2) = T(r)T(s)$ or equivalently $r \circ r = s \circ s = r \circ s$, because $T$ is injective. So, by theorem 5.8 [8], $\ell^1(S)$ is semisimple if and only if $\ell^1(S_T)$ is semisimple.

**Theorem 3.2.** Let $S$ be a discrete semigroup and $T \in Mul_l(S)$. Then
(i) The left multiplier $T$ has an extension $\overline{T} \in Mul_l(\ell^1(S))$ with the norm decreasing.
(ii) The left multiplier $T$ is injective if and only if so is $\overline{T}$.
(iii) If $T$ is injective then $\overline{T}$ is an isometry and also $\ell^1(S_T)$ and $\ell^1(S_T)$ are isomorphic.

**Proof.** (i) An arbitrary element $f \in \ell^1(S)$ is of the form $f: S \to \mathbb{C}$ such that $f(x) = 0$ except at the most countable subset $A$ of $S$. If $A$ is a finite subset of $S$ then $f = \sum_{k=1}^{n} f(x_k) \delta_{x_k}$ for some fixed $n \in \mathbb{N}$. So in general we have
\[
f = \sum_{x \in S} f(x) \delta_x = \sum_{x \in A} f(x) \delta_x + \sum_{k=1}^{n} f(x_k) \delta_{x_k}.
\]
Now, for each $n \in \mathbb{N}$, let $f_n = \sum_{k=1}^{n} f(x_k) \delta_{x_k}$ and define $\overline{T}: \ell^1(S) \to \ell^1(S)$ by

\[
\overline{T}(\delta_x) = \delta_T(x) \quad (x \in S),
\]
\[
\overline{T}(f_n) = \sum_{k=1}^{n} f(x_k) \overline{T}(\delta_{x_k}) = \overline{f_n}.
\]

For each $m, n \in \mathbb{N}$ where $n \geq m$, we have
\[
\|\overline{T}(f_n) - \overline{T}(f_m)\|_1 = \|\overline{f_n} - \overline{f_m}\|_1 = \|\sum_{k=m}^{n} f(x_k) \overline{T}(\delta_{x_k})\| = \|\sum_{k=m}^{n} f(x_k) \delta_{T(x_k)}\|
\leq \sum_{k=m}^{n} |f(x_k)| = \|f_n - f_m\|_1.
\]
So \( \{ \tilde{T}(f_n) \}_n \) is a Cauchy sequence and it is convergent. Now, we define \( \tilde{T}(f) = \lim_n \tilde{T}_n \). Then the definition is well defined. Hence
\[
\tilde{T}(f) = \sum_{k=1}^{\infty} f(x_k) \delta_{x_k} = \hat{f},
\]
also
\[
\| \tilde{T}(f) \|_1 \leq \sum_{x \in \mathcal{A}} |f(x_k)| = \| f \|_1 \quad \text{or} \quad \| \tilde{T}(f) \|_1 \leq \| f \|_1.
\]
It shows that \( \tilde{T} \) is norm decreasing.

In the following, we extend \( \tilde{T} \) by linearity. Let \( f, g \in \ell^1(\mathcal{S}) \). Then there are two at most countable sub set \( A, B \) of \( \mathcal{S} \) such that
\[
f = \sum_{x \in A} f(x) \delta_x, \quad g = \sum_{x \in B} g(x) \delta_x.
\]
Suppose that \( D = A \cup B \). So we have \( f + g = \sum_{x \in D} (f(x) + g(x)) \delta_x \).

Then, it follows that
\[
\tilde{T}(f + g) = \tilde{T} \left( \sum_{x \in D} (f(x) + g(x)) \delta_x \right) = \sum_{x \in A} \tilde{T}(f(x)) \tilde{T}(\delta_x) + \sum_{x \in B} \tilde{T}(g(x)) \tilde{T}(\delta_x) = \hat{f} + \hat{g}.
\]
Also, if \( \alpha \in \mathbb{C} \), we have
\[
\tilde{T}(\alpha f) = \alpha \hat{f} = \sum_{x \in \mathcal{A}} \alpha f(x) \tilde{T}(\delta_x) = \alpha \sum_{x \in \mathcal{A}} f(s) \tilde{T}(\delta_x) = \alpha \hat{f}.
\]
Therefore, \( \tilde{T} \) is a bounded linear isometry.

Now, we prove that \( \tilde{T} \in \text{Mult}(\ell^1(\mathcal{S})) \). Take \( x, y \in \mathcal{S} \). Then
\[
\tilde{T}(\delta_x * \delta_y) = \tilde{T}(\delta_{xy}) = \delta_{T(xy)} = \delta_{T(x)y} = \delta_{T(x)} * \delta_y = \tilde{T}(\delta_x) * \delta_y.
\]
Let \( y \in \mathcal{S} \) be fixed and \( f, g \in \ell^1(\mathcal{S}) \). Then
\[
\tilde{T}(f * \delta_y) = \tilde{T} \left( \sum_{x \in \mathcal{A}} f(x) \delta_{xy} \right) = \sum_{x \in \mathcal{A}} f(x) \tilde{T}(\delta_{xy}) = \left( \sum_{x \in \mathcal{A}} \tilde{T}(\delta_x) \right) * \delta_y = \hat{f} * \delta_y = \tilde{T}(f) * \delta_y.
\]
In the general case, we have
\[
\tilde{T}(f * g) = \tilde{T} \left( \sum_{x \in \mathcal{A}} f(x) (\sum_{y \in \mathcal{B}} g(y) \delta_{xy}) \delta_x \right) = \sum_{x \in \mathcal{A}} f(x) \sum_{y \in \mathcal{B}} g(y) \tilde{T}(\delta_x) * \delta_y
\]
\[
= \sum_{x \in \mathcal{A}} f(x) \tilde{T}(\delta_x) * \sum_{y \in \mathcal{B}} g(y) \delta_y = \tilde{T}(f) * g.
\]
This shows that \( \tilde{T} \) is a multiplier on \( \ell^1(\mathcal{S}) \).

(ii) Let \( T \) be injective. Take \( x, y \in \mathcal{S} \) and suppose that \( \tilde{T}(\delta_x) = \tilde{T}(\delta_y) \). Then \( \delta_{T(x)} = \tilde{T}(\delta_x) = \tilde{T}(\delta_y) = \delta_{T(y)} \).

Therefore, \( T(x) = T(y) \). Since \( T \) is injective, we have \( x = y \). It follows that \( \delta_x = \delta_y \), consequently \( \tilde{T} \) is injective.

Conversely, the same argument shows that the converse holds.

(iii) Let \( T \) be injective and \( f \in \ell^1(\mathcal{S}) \). Then there exists at most a countable subset \( A \subseteq \mathcal{S} \) such that
Since $A$ and $T(A)$ have the same cardinal number, \[ \|\bar{T}(f)\| = \|\sum_{x \in A} f(x) \delta_x\| = \sum_{x \in A}|f(x)| = \|f\|_1 , \]
so $\bar{T}$ is an isometry. Now, we can define a new multiplication "\( \boxplus \)" on $\ell^1(S)$ as follows \[ f \boxplus g = f * \bar{T}g \quad (f, g \in \ell^1(S)) \]. By a similar argument in theorem 1.3.1 [10], $\ell^1(S)$ with the new product is a Banach algebra that is denoted it by $\ell^1(S)_{\bar{T}}$. We define the map $\Psi: \ell^1(S_T) \to \ell^1(S)_{\bar{T}}$, by \[ \Psi(\delta_x) = \delta_x \quad (x \in S) \].

Take $x, y \in S$. Then \[
\Psi(\delta_x * \delta_y) = \Psi(\delta_{x+y}) = \delta_{xT(y)} = \delta_x * \delta_{T(y)} = \delta_x * \bar{T}(\delta_y) = \delta_x \boxplus \delta_y = \Psi(\delta_x) \boxplus \Psi(\delta_y).
\]
So, in general case, we have \[ \Psi(f * g) = \Psi(f) \boxplus \Psi(g) \quad (f, g \in \ell^1(S)) \]. Thus, $\Psi$ is an isomorphism. Therefore $\ell^1(S_T)$ and $\ell^1(S)_{\bar{T}}$ are isomorphic.

**Theorem 3.3.** Let $S$ be a semigroup and $T \in \text{Mul}_f(S)$ be bijective. Then $\ell^1(S)$ is amenable if and only if $\ell^1(S_T)$ is amenable.

**Proof.** By theorem 3.2, we have $\ell^1(S_T) \cong \ell^1(S)_{\bar{T}}$. Suppose that $\ell^1(S_T)$ is amenable and define $\varphi: \ell^1(S)_{\bar{T}} \to \ell^1(S)_{\bar{T}}$ by $\varphi(f) = \bar{T}(f)$. Take $x, y \in S$. Then \[
\varphi(\delta_x \boxplus \delta_y) = \bar{T}((\delta_x \boxplus \delta_y)) = \bar{T}(\delta_{xT(y)}) = \bar{T}(\delta_x * \delta_{T(y)}) = \bar{T}(\delta_x) * \bar{T}(\delta_y) = \bar{T}(\delta_x) \boxplus \bar{T}(\delta_y).
\]
Now, by induction and continuity of $\bar{T}$, we have \[ \varphi(f \boxplus g) = \varphi(f) \boxplus \varphi(g). \] If $T$ is bijective, $\bar{T}$ is bijective. Therefore $\varphi$ is an epimorphism of $\ell^1(S_T)$ onto $\ell^1(S)$.

Hence, by proposition 2.3.1 [16], $\ell^1(S)$ is amenable.

Conversely, suppose that $\ell^1(S)$ is amenable. Since $T$ is bijective, $\bar{T}$ is bijective. Therefore $\bar{T}^{-1}$ exists. Now define $\theta: \ell^1(S) \to \ell^1(S_T) \cong \ell^1(S)_{\bar{T}}$ by $\theta(f) = \bar{T}^{-1}(f)$.

Take $x, y \in S$. Then \[
\theta(\delta_x * \delta_y) = \bar{T}^{-1}(\delta_{xy}) = \bar{T}^{-1}(\delta_x) \bar{T}^{-1}(\delta_y) = \bar{T}^{-1}(\delta_x) \boxplus \bar{T}^{-1}(\delta_y) = \theta(\delta_x) \boxplus \theta(\delta_y).
\]
Similarly $\theta$ is an epimorphism from $\ell^1(S)$ onto $\ell^1(S_\tau)$. By proposition 2.3.1 [16] $\ell^1(S_\tau)$ is amenable.

Note that, in general, it is not known when $\ell^1(S)$ is weakly amenable. For more details see [2].

**Theorem 3.4.** Let $S$ be a semigroup and $T \in \text{Mul}_l(S)$ be bijective. Then, if $S$ is completely regular then $\ell^1(S_\tau)$ is weakly amenable.

**Proof.** It is enough to prove that $S_\tau$ is completely regular, then by theorem 3.6 [2], $\ell^1(S_\tau)$ can be weakly amenable. Take $s \in S$. Then there exists $r \in S$ such that $T(s) = T(s)T(r)T(s)$, $T(r)T(s) = T(s)T(r)$, since $T$ is bijective and $S = T(S)$ is completely regular. So we have $T(s) = T(s \circ r \circ s)$ and $T(r \circ s) = T(s \circ r)$. Hence $s = s \circ r \circ s$ and $r \circ s = s \circ r$ for some $r \in S$, since $T$ is injective. Therefore $S_\tau$ is completely regular.

**Corollary 3.5.** Suppose that $S$ is a commutative completely regular semigroup and $T \in \text{Mul}_l(S)$ is injective. Then $\ell^1(T(S)_\tau)$ is weakly amenable.

**Proof.** [2, theorem 3.6 ] $\ell^1(S)$ is weakly amenable. Define $\varphi : S \to \ell^1(S)_\tau$ by

$$\varphi(s) = T^{-1}(s) \quad (s \in S).$$

We show that $\varphi$ is a homomorphism. Take $s \in S$, then we have

$$\varphi(rs) = T^{-1}(rs) = T^{-1}(r)s = T^{-1}(r) \circ (T^{-1}s).$$

So $\varphi$ is a homomorphism. Then by proposition 2.1[7], $\ell^1(T(S)_\tau)$ is weakly amenable. In the case that $S$ is a group, it is easy to see that the amenability of $S$ implies the amenability of $\ell^1(S_\tau)$. Indeed, when $S$ is a group, by theorem 2.1, $S_\tau$ is a semigroup and one can easily prove that $S_\tau$ is also a group. On the other hand, $\text{Mul}_l(S) \cong S$ because $S$ is a unital semigroup, so each $T \in \text{Mul}_l(S)$ is inner and of the form $T = L_s$ for some $s \in S$. Also $T^{-1} = L_{a^{-1}}$ exists, since $S$ is a group. Then the map $\theta : S_\tau \to S$ defined by $\theta(s) = T(s)$ is an isomorphism; that is $S \cong S_\tau$. Thus we have the following result:

**Corollary 3.6.** Let $S$ be a cancellative regular discrete semigroup. Then $\ell^1(S)$ is amenable if and only if $\ell^1(S_\tau)$ is amenable.

**Proof.** By [9, Exercise 2.6.11] $S$ is a group. So the assertion holds by [15, theorem 2.1.8]
Examples

In this section we present some examples which either comment on our results or indicate necessary condition in our theorems.

4.1. There are semigroups \( S \) and \( T \in \text{Mult}(S) \) such that the background semigroups \( S \) are not commutative but their induced semigroups \( S_T \) are commutative.

This example shows that the condition \( T(S) = S \), in theorem 2.2, can not be omitted.

Let \( S \) be the set \{a, b, c, d, e\} with operation table given by

\[
\begin{array}{cccccc}
  & a & b & c & d & e \\
  a & a & a & a & d & d \\
b & a & b & c & d & d \\
c & a & c & b & d & d \\
d & d & d & d & a & a \\
e & d & e & e & a & a \\
\end{array}
\]

Clearly \((S, .)\) is a non-commutative semigroup. Now, put \( T = L_a \) where \( L_a(x) = ax \) for all \( x \in S \). One can get easily the operation table of \( S_T \) as follow:

\[
\begin{array}{cccccc}
  & a & b & c & d & e \\
  a & a & a & a & d & d \\
b & a & a & a & d & d \\
c & a & a & a & d & d \\
d & d & d & d & a & a \\
e & d & d & d & a & a \\
\end{array}
\]

The operation table shows that the induced semigroup \( S_T \) is commutative and \( T(S) \neq S \). Also the other induced semigroup \( S_T \) is commutative for \( T = L_d \) analogously.

Now we present some important theorems from[14] that we need in the following examples:

**Theorem 4.2.** Let \( S \) be a semigroup. Suppose that \( t^1(S) \) is amenable. Then

(i) \( S \) is amenable

(ii) \( S \) is regular.

(iii) \( E(S) \) is finite.

(iv) \( t^1(S) \) has an identity.

**Proof.** (i) That is lemma 3 in [5].

(ii) and (iii) See theorem 2 in[6].
(iv) That is corollary 10.6 in [4].

**Theorem 4.3.** Let $S$ be a finite semigroup. Then the following statements are equivalent:

(i) $\ell^1(S)$ is amenable.
(ii) $S$ is regular and $\ell^1(S)$ is nuital.

Proof. Refer to [3].

4.4. There are semigroups $S$ and $T \in \text{Mul}(S)$ such that $S$ and $\ell^1(S)$ are amenable but $S_T$ is not regular and also, $\ell^1(S_T)$ is not amenable.

This example shows that two semigroup algebras $\ell^1(S)$ and $\ell^1(S_T)$ can be different in some properties. Also, it notifies that the bijectivity of $T$ in the theorem 3.3 is essential. Put $S = \{x_0, x_1, x_2, \ldots, x_n\}$ with the operation $x_i x_j = x_{\text{Max}(i,j)}$ ($0 \leq i, j \leq n$, $n \geq 2$).

Then $S$ is a semigroup. Since

$$\text{Max}\{i, \text{Max}\{j, k\}\} = \text{Max}\{\text{Max}\{i, j\}, k\} = \text{Max}\{i, j, k\}.$$  

We denote it by $S_v$. This semigroup is commutative. So by (0.18) in [12], it is amenable. $S_v$ is a unital semigroup and has a zero; indeed, $e_s = x_0$ and $a_s = x_n$. Also, it is a regular semigroup and $\text{Mul}(S_v) \cong S_v$ because $S_v$ has an identity.

Evidently, $S_v$ is regular since each $s \in S_v$ is idempotent. The semigroup algebra $\ell^1(S_v)$ is a unital algebra because $S_v$ has an identity. So by theorem 4.3 (ii) $\ell^1(S_v)$ is amenable.

Now, take $T = L_{x_k}$ for a fixed $x_k \in S$ where $k \geq 1$. By theorem 2.2, $(S_v)_T$ is commutative so is amenable. We show that $T$ is neither injective and nor surjective.

Take $x_i \in S_v$, then $Tx_i = x_k x_i = x_{\text{Max}(k,i)}$. So

$$T(S_v) = \{x_k, x_{k+1}, \ldots, x_n\} \neq S_v.$$  

Hence, $T$ is not surjective.

Again, take distinct elements $x_i, x_j$ in $S_v$ for some $i, j < k$ such that $T(x_i) = T(x_j)$.

Then we have $x_{\text{Max}(k,i)} = x_{\text{Max}(k,j)}$ but $x_i \neq x_j$. So $T$ is not injective.

We prove that $(S_v)_T$ is not regular. If $(S_v)_T$ is regular, then for $x_{k-1} \in S_v$ there exists an element $x_j \in S_v$ such that

$$x_{k-1} = x_{k-1} \circ x_j \circ x_{k-1} = x_{\text{Max}(k,j)}.$$
That implies that \( \max\{k,j\} = k - 1 \); which is impossible. Consequently, by theorem 4.2 (ii) or 4.3 (ii), \( l^1(\mathcal{S}_\gamma T) \) is not amenable.

Also, the inequality \( \mathcal{S}_\gamma \circ \mathcal{S}_\gamma = \{x_k, x_{k+1}, \ldots, x_n\} \neq \mathcal{S}_\gamma \) shows that \( l^1(\mathcal{S}_\gamma T) \) is not weakly amenable. In the next example we show that in the theorem 3.2 (iii) the condition "injectivity of " can not be omitted.

**4.5** There are a semigroup \( \mathcal{S} \) and \( T \in Mul_l(\mathcal{S}) \) such that \( T \in Mul_l(\mathcal{S}) \) is not injective and the corresponding \( \tilde{T} \in Mul_l(\mathcal{S}_T) \) is not an isometry.

Suppose that \( \mathcal{S}_\gamma \) is a semigroup as in example 4.4 and \( T = L_{x_k} \) for some fixed \( 1 < k < n \). If \( f \in \mathcal{S}_\gamma \) then \( f = \sum_{i=0}^n f(x_i) \delta_{x_i} \) and also \( \mathcal{T}(f) = \sum_{i=0}^n f(x_i) \delta_{T(x_i)} \). But
\[
T(x_i) = \begin{cases} x_i & k < i \leq n \\ x_k & 0 \leq i \leq k \end{cases},
\]
so
\[
\mathcal{T}(f) = \left( \sum_{i=0}^k f(x_i) \right) \delta_{x_k} + \sum_{i=k+1}^n f(x_i) \delta_{T(x_i)}.
\]

Hence
\[
\| \tilde{T}(f) \| = \left| \sum_{i=0}^k f(x_i) \right| + \sum_{i=k+1}^n |f(x_i)|
\leq \sum_{i=0}^k |f(x_i)| + \sum_{i=k+1}^n |f(x_i)| = \| f \|_1.
\]

It shows that \( \tilde{T} \) is not an isometry.

**4.6.** There are semigroups \( \mathcal{S} \) and \( T \in Mul_l(\mathcal{S}) \) such that \( l^1(\mathcal{S}) \) is semisimple. But \( l^1(\mathcal{S}_T) \) is not semisimple. This example remind that, in theorem 3.1 the multiplier \( T \) must be injective.

Let \( \mathcal{S} \) be a set \( \{x_0, x_1, \ldots, x_n\} \) where \( n \in \mathbb{N} \) and \( n \geq 3 \) is fixed. by operation given by \( xy = x_{\min\{i,j\}} \), \( \mathcal{S} \) is a commutative semigroup. Since
\[
\min\{i, \min\{j,k\}\} = \min\{\min\{i,j\}, k\} = \min\{i,j,k\} \quad (i,j,k \in \mathbb{N}).
\]

We denote it briefly by \( \mathcal{S}_\wedge \). For each \( x, y \in \mathcal{S} \) the equality \( x^2 = y^2 = xy \) implies \( x = y \). So by Theorem 5.8 [8] \( l^1(\mathcal{S}_\wedge) \) is semisimple.

Now, let \( T = L_{x_k} \) for a fixed \( 1 \leq k < n - 1 \). It is easy to see that \( T(x_k) = T(x_n) \) but \( x_k \neq x_n \). So the multiplier \( T \) is not injective.

We show that neither \( \mathcal{S}_\wedge \) nor \( l^1(\mathcal{S}_\wedge)_T \) is semisimple.

Each ideal of \( \mathcal{S} \) is of the form
\[
I_m = \{x_0, x_1, \ldots, x_m\} \quad (m \leq n).
\]

We claim that \( \mathcal{S}_T \) is not semisimple. Since for each \( m \in \mathbb{N} \) we have
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\[
I_m \circ I_m = \begin{cases} 
I_m & m \leq k \\
I_k & m > k 
\end{cases}
\]

On the other hand, for each $x_i, x_j \in S$ where $i \neq j$ and $i, j > k$, we have $x_i \circ x_i = x_j \circ x_j = x_i \circ x_j = x_k$, while $x_i \neq x_j$. Thus, Theorem 5.8 [8] shows that $\ell^1(S_n)_T$ is not semisimple.

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Reference


