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A numerical Algorithm Based on Chebyshev Polynomials for Solving some Inverse Source Problems

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Abstract

In this paper, two inverse problems of determining an unknown source term in a parabolic equation are considered. First, the unknown source term is estimated in the form of a combination of Chebyshev functions. Then a numerical algorithm based on Chebyshev polynomials is presented for obtaining the solution of the problem. For solving the problem, the operational matrices of integration and derivation are introduced and utilized to reduce the mentioned problem into the matrix equations which correspond to a system of linear algebraic equations with unknown Chebyshev coefficients. Due to ill-posedness of these inverse problems, the Tikhonov regularization method with generalized cross validation (GCV) criterion is applied to find stable solutions. Finally, some examples are presented to illustrate the efficiency of this numerical method. The numerical results show that the proposed method is a reliable method and can give high accuracy approximate solutions.

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Introduction

Inverse problems play an important role in various fields of science and engineering which aim to detect some unknown parameters from some additional data related to the problem. These kinds of problems have been studied by many authors [1-5].

Consider the following problem of determining the function $u(x,t)$ satisfying the parabolic equation

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$$u_t - u_{xx} = s(x,t), \qquad 0 < x < L,\ 0 < t < \tau, \tag{1}$$

with the initial condition

$$u(x,0) = u_0(x), \qquad 0 < x < L, \tag{2}$$

and the boundary conditions

$$u(0,t) = g_0(t), \quad u(L,t) = g_1(t), \quad 0 < t < \tau, \tag{3}$$

where $u_0(x)$, $g_0(t)$ and $g_1(t)$ are piecewise continuous functions in their domains. Also these functions satisfy the conditions $u_0(0) = g_0(0)$ and $u_0(1) = g_1(0)$. This problem is induced in the process of transportation, diffusion and conduction of natural materials [6,7].

Here we consider that the source term, s(x,t), is an unknown function which needs to be determined. This problem is called as inverse source problem [1]. In many applications resulting the parabolic type diffusion problems, the source terms are usually not easy to detect directly. For example, the problem of identifying sources of water and air pollution intensity in the environment is a case of this type of problem. Solute transport in a uniform groundwater flow can be described by the parabolic equation

$$\bar{u}_t - D\bar{u}_{xx} + V\bar{u}_x + R\bar{u} = s_0(x,t), \quad x \in \Omega, 0 < t < T,$$

where Ω is a spatial domain, $\bar{u}$ is the solute concentration, V represents the velocity of watershed movement, R denotes the self-purifying function of the watershed, and $s_0(x,t)$ is a source term causing the pollution function $\bar{u}(x,t)$ [8]. Putting

$$\bar{u}(x,t) = u(x,t)e^{\frac{V}{2D}x - (\frac{V^2}{4D}+R)t},$$

we get

$$u_t - Du_{xx} = s(x,t),$$

Where

$$\mathrm{s}(x,t) = s_0(x,t)e^{-\frac{V}{2D}x + (\frac{V^2}{4D}+R)t}.$$

An estimation of pollutant source is vital to environmental safeguard in cities with high populations [9]. As another example, heat source identification problems are the most commonly encountered inverse problems in heat conduction.

In this study, we suppose that source function is separable and can be represented in the form

$$s(x,t) = f(x)g(t), \qquad 0 < x < L,\ 0 < t < \tau, \tag{4}$$

where one of the functions f(x) or g(t) is known in its domain and the other function is unknown which remains to be determined. Many researchers have used this assumption to estimate unknown source function, in literature [10-28]. In different applications, one of these terms assumes to be specified according to the physical and environmental conditions of the problem and the other term will be approximated to find the unknown source function $s\left(x,t\right)$. For solving this type of problems, an overspecified condition shall be considered. Therefore, we define two problems as follows:

Problem 1. Consider the equation (1) with the source term function as (4). Suppose $g(t)$ is given and the function $f(x)$ is unknown. For solving this inverse problem, we assume an overspecified condition as the final data

$$u(x,\tau)=\varphi(x), \quad 0<x<L. \tag{5}$$

Problem 2. In this case, we consider the equation (1) with the source term function as (4) where the function $g(t)$ is unknown. We suppose the overspecified condition as

$$u(x_0,t)=\chi(t), \quad 0<t<\tau, \tag{6}$$

for solving this inverse problem, where $< x_0 < L$ is a constant.

The overspecified conditions (5) and (6) are necessary for unique solvability of these two inverse problems. The unique solutions for these problems are discussed in [10, 11]. In the case of heat source identification problems, the condition (5) is the additional temperature measurement at a terminal moment of time τ and the condition (6) is the overspecified temperature values at an interior point $x_0 \in (0,L)$ where a thermocouple is located to record the temperature measurement.

Problem 1 and Problem 2 belong to the class of inverse source problems, which are ill-posed [1, 12-17], since small errors in any practical input data, give rise to unbounded and highly oscillatory solutions. So, numerical reconstruction is very difficult and some special regularization methods are required to obtain an accurate approximation solution. In [18-23] researchers have investigated some numerical methods for special cases of reconstructing the space-dependent source term. Also for the case of recovering the time-dependent source term, authors in [23-28] have introduced some other numerical schemes.

In the rest of this paper, by using the conditions (5) and (6) as overspecified data, a numerical algorithm is presented for solving these two inverse problems based on the

Chebyshev polynomials. The paper is organized as follows: In Section 2, basic properties of bivarniiate second kind Chebyshev polynomials are presented and operational matrices of these polynomials are introduced. In Section 3, we give an approximation of the unknown source function. Also, the approximate solution of the main problem is presented in Section 4. In Section 5, some numerical examples are presented. The conclusion is included in Section 6.

Preliminaries

2.1. Definition and function approximation

The second kind Chebyshev polynomials (SKCPs) are orthogonal polynomials on the interval $[-1,1]$ and can be determined with the following recurrence formula [29]:

$$U_{i+1}(x) = 2xU_i(x) - U_{i-1}(x), \quad i = 1,2,3,\ldots,$$

where $U_0(x) = 1$ and $U_1(x) = 2x$. By the change of variable we will have the well-known shifted SKCPs on the interval $[0, L]$ as follows

$$U_{L,i}(x) = U_i(\frac{2}{L}x - 1), \; i = 0,1,2,\cdots.$$

The function $U_{L,i}(x)$ satisfies the following properties

$$U_{L,i}(0) = (-1)^i (i+1), \tag{7}$$

$$U_{L,i}(L) = i+1, \tag{8}$$

$$U'_{L,i}(x) = \frac{4}{L}\sum_{k=0}^{[i/2]}(i-2k)U_{L,i-2k-1}(x), \; U_{L,-1}(x) = 0, \tag{9}$$

$$\int_0^x U_{L,i}(s)ds = \frac{L}{2(i+1)}[(-1)^i U_{L,0}(x) - \frac{1}{2}U_{L,i-1}(x) + \frac{1}{2}U_{L,i+1}(x)], \; U_{L,-1}(x) = 0. \tag{10}$$

The orthogonality condition is

$$\int_0^L w_L(x)U_{L,i}(x)U_{L,j}(x)dx = \frac{L\pi}{4}\delta_{ij},$$

where

$$w_L(x) = \sqrt{1-(\frac{2}{L}x-1)^2}.$$

A function $f(x)$, square integrable on $[0, L]$, may be expanded in terms of the shifted SKCPs as follows

$$f(x) = \sum_{i=0}^{\infty} a_i U_{L,i}(x),$$

where the coefficients a_i are given by

$$a_i = \frac{4}{L\pi}\int_0^L w_L(x) f(x) U_{L,i}(x)dx, \quad i = 0,1,2,\cdots. \tag{10}$$

To obtain the coefficients a_i, we use the second kind Gauss-Chebyshev quadrature formula [29] as

$$a_i = \frac{2}{K+1}\sum_{k=1}^{K} sin^2(\frac{k\pi}{K+1}) f(x_k) U_{L,i}(x_k),$$

where x_k, $k = 1,2,\dots,K$ are the zeroes of $U_{L,K}(x)$ as

$$x_k = \frac{L}{2}(cos(\frac{k\pi}{K+1})+1). \tag{11}$$

In practice, only the first (N+1)-terms are considered. Hence, $f(x)$ can be approximated as follows

$$f(x) \simeq \sum_{i=0}^{N} a_i U_{L,i}(x) = A^T \psi_L(x) = \psi_L^T(x) A, \tag{12}$$

where the coefficient vector A and the shifted Chebyshev vector $\psi_L(x)$ are given by

$$A = [a_0, a_1, \dots, a_N]^T,$$

$$\psi_L(x) = [U_{L,0}(x), U_{L,1}(x), \dots, U_{L,N}(x)]^T. \tag{13}$$

Definition 2.1. Bivariate shifted SKCPs are defined on $[0,L]\times[0,\tau]$ as

$$\phi_{ij}^{L,\tau}(x,t) = U_{L,i}(x) U_{\tau,j}(t), \quad i,j = 0,1,2,\cdots. \tag{14}$$

Bivariate shifted SKCPs are orthogonal with each other as:

$$\int_0^\tau \int_0^L w_{L,\tau}(x,t) \phi_{ij}^{L,\tau}(x,t) \phi_{mn}^{L,\tau}(x,t) dxdt = \frac{L\tau\pi^2}{16}\delta_{im}\delta_{jn},$$

where $w_{L,\tau}(x,t) = w_L(x) w_\tau(t)$.

A function $u(x,t)$ defined on $[0,L]\times[0,\tau]$ may be approximated in terms of the bivariate SKCPs as follows

$$u(x,t) \simeq \sum_{i=0}^{N}\sum_{j=0}^{N} c_{ij}\phi_{ij}^{L,\tau}(x,t) = C^T \Phi_{L,\tau}(x,t) = \Phi_{L,\tau}^T(x,t) C, \tag{15}$$

where

$$C = [c_{00}, c_{01}, \dots, c_{0N}, \dots, c_{N0}, c_{N1}, \dots, c_{NN}]^T,$$

$$\Phi_{L,\tau}(x,t) = \psi_L(x) \otimes \psi_\tau(t)$$
$$= [\phi_{00}^{L,\tau}(x,t), \phi_{01}^{L,\tau}(x,t), \dots, \phi_{0N}^{L,\tau}(x,t), \dots, \phi_{N0}^{L,\tau}(x,t), \phi_{N1}^{L,\tau}(x,t), \phi_{NN}^{L,\tau}(x,t)]^T. \tag{17}$$

and $\otimes$ denotes the Kronecker product defined for two arbitrary matrices A and B as (for more details see [30], Chapter 13):

$$A \otimes B = (a_{ij} B).$$

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The coefficients c_{ij} in equation (16) are obtained as follows

$$c_{ij} = \frac{16}{L\tau\pi^2}\int_0^\tau\int_0^L w_{L,\tau}(x,t)u(x,t)\phi_{ij}^{L,\tau}(x,t)dxdt.$$

The following second kind Gauss-Chebyshev quadrature formula is used to obtain the coefficients c_{ij} ,

$$c_{ij} = \frac{4}{(K+1)^2}\sum_{k=1}^{K}\sum_{l=1}^{K} sin^2(\frac{k\pi}{K+1})sin^2(\frac{l\pi}{K+1})u(x_k,t_l)\phi_{ij}^{L,\tau}(x_k,t_l),$$

where x_k is defined by (12) and t_l , $l = 1,2,\ldots,K$ are the zeroes of $U_{\tau,K}(t)$ as

$$t_l = \frac{\tau}{2}(cos(\frac{l\pi}{K+1})+1).$$

2.2. Operational matrices

In this section, we will use the properties of the shifted SKCPs introduced above, to construct the needed operational matrices.

Theorem 2.1. Let $\Phi_{L,\tau}(x,t)$ be the bivariate shifted Chebyshev vector given by (17), then

$$\frac{\partial\Phi_{L,\tau}(x,t)}{\partial x} = D\Phi_{L,\tau}(x,t), \tag{16}$$

where D is an $(N+1)^2 \times (N+1)^2$ matrix as

$$D = \frac{4}{L}\begin{bmatrix} O & O & O & O & \cdots & O & O \\ I & O & O & O & \cdots & O & O \\ O & 2I & O & O & \cdots & O & O \\ I & O & 3I & O & \cdots & O & O \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ M_1 & M_2 & M_3 & M_4 & \cdots & NI & O \end{bmatrix},$$

in which M_1, M_2, M_3 and M_4 are I , O , $3I$ and O , for odd N and O , $2I$, O and $4I$, for even N, respectively, and I and O are the identity and zero matrix of order N+1, respectively.

Proof. Using the definition of the vector $\Phi_{L,\tau}(x,t)$ we get

$$\frac{\partial\Phi_{L,\tau}(x,t)}{\partial x} = \frac{\partial(\psi_L(x)\otimes\psi_\tau(t))}{\partial x} = \psi'_L(x)\otimes\psi_\tau(t). \tag{19}$$

On the other hand, using equation (9) we have

$$U'_{L,0}(x) = \frac{4}{L}[0,0,0,0,....,0,0]\psi_L(x),$$

$$U'_{L,1}(x) = \frac{4}{L}[1,0,0,0,....,0,0]\psi_L(x),$$

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$$U'_{L,2}(x)=\frac{4}{L}[0,2,0,0,....,0,0]\psi_L(x),$$
$$U'_{L,3}(x)=\frac{4}{L}[1,0,3,0,....,0,0]\psi_L(x),$$
$$\vdots$$

Finally, for even N we obtain

$$U'_{L,N}(x)=\frac{4}{L}[0,2,0,4,....,N,0]\psi_L(x),$$

and for odd N we get

$$U'_{L,N}(x)=\frac{4}{L}[1,0,3,0,....,N,0]\psi_L(x).$$

Therefore, we have

$$\psi'_L(x)=D_0\psi_L(x), \tag{20}$$

where

$$D_0=\frac{4}{L}\begin{bmatrix} 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 2 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 3 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ m_1 & m_2 & m_3 & m_4 & \cdots & N & 0 \end{bmatrix},$$

in which m_1, m_2, m_3 and m_4 are 1, 0, 3 and 0, for odd N and 0, 2, 0 and 4, for even N, respectively.

Substituting (20) into (19) and using the properties of the Kronecker product yield

$$\frac{\partial\Phi_{L,\tau}(x,t)}{\partial x}=(D_0\psi_L(x))\otimes(I\,\psi_\tau(t))=(D_0\otimes I)(\psi_L(x)\otimes\psi_\tau(t))=D\Phi_{L,\tau}(x,t),$$

which completes the proof. □

In a similar way, it can be proved using equation (10), that the integration of the vector $\Phi_{L,\tau}(x,t)$ with respect to t can be approximated by

$$\int_0^t \Phi_{L,\tau}(x,t')dt' \simeq P\Phi_{L,\tau}(x,t), \tag{21}$$

where P is an $(N+1)^2\times(N+1)^2$ matrix as

$$P=\frac{\tau}{2}\begin{bmatrix} Q & O & O & \cdots & O \\ O & Q & O & \cdots & O \\ O & O & Q & \cdots & O \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ O & O & O & \cdots & Q \end{bmatrix},$$

and

$$Q=\begin{bmatrix} 1 & \frac{1}{2} & 0 & 0 & 0 & \dots & 0 & 0 \\ \frac{-3}{4} & 0 & \frac{1}{4} & 0 & 0 & \dots & 0 & 0 \\ \frac{1}{3} & \frac{-1}{6} & 0 & \frac{1}{6} & 0 & \dots & 0 & 0 \\ \frac{-1}{4} & 0 & \frac{-1}{8} & 0 & \frac{1}{8} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ \frac{(-1)^N}{N+1} & 0 & 0 & 0 & 0 & \dots & \frac{-1}{2(N+1)} & 0 \end{bmatrix}.$$

To suggest a numerical algorithm, we need to introduce two more operational matrices W_0 and W_1, such that

$$\Phi_{L,\tau}(0,t)=W_0^T\psi_\tau(t)=\psi_\tau^T(t)W_0, \tag{22}$$

$$\Phi_{L,\tau}(L,t)=W_1^T\psi_\tau(t)=\psi_\tau^T(t)W_1, \tag{23}$$

where $\psi_\tau(t)$ is defined by (14) and W_0 and W_1 are two $(N+1)\times(N+1)^2$ matrices which can be obtained using equation (7) and (8), respectively, as follows

$$W_0=\begin{bmatrix} I & -2I & 3I & \dots & (-1)^N(N+1)I \end{bmatrix},$$

$$W_1=\begin{bmatrix} I & 2I & 3I & \dots & (N+1)I \end{bmatrix}.$$

Determination of the source function

In this section, we give an approximation of the unknown source function using the shifted SKCPs. To this aim, we transform problem (1)-(3) to a zero initial and boundary conditions problem with the change of variables as follows

$$w(x,t)=u(x,t)-e^{-t}u_1(x)-(g_1(t)-g_0(t))x-g_0(t), \tag{24}$$

where

$$u_1(\mathrm{x})=u_0(\mathrm{x})-(\mathrm{g}_1(0)-\mathrm{g}_0(0))x-\mathrm{g}_0(0).$$

Therefore, problem (1)-(3) is equivalent to the following problem

$$\begin{aligned} &w_t(x,t)-w_{xx}(x,t)=s(x,t)+F(x,t), \quad 0<x<L, 0<t<\tau, \\ &w(x,0)=0, \quad 0<x<L, \\ &w(0,t)=0, \quad w(L,t)=0, \quad 0<t<\tau, \end{aligned}$$

where

$$F(x,t)=e^{-t}u_0''(x)+e^{-t}(u_0(x)-(g_1(0)-g_0(0))x-g_0(0))-(g_1'(t)-g_0'(t))x-g_0'(t).$$

By using the separation of variables, the solution of this problem may be expressed as follows

$$w(x,t)=2\sum_{k=1}^{\infty}\left(\int_0^t\int_0^L\left(s(\xi,\eta)+F(\xi,\eta)\right)\sin(k\pi\xi)e^{-(k\pi)^2(t-\eta)}d\xi d\eta\right)\sin(k\pi x), \quad (25)$$

Considering equations (5), (6) and (25), we obtain the following results.

Problem 1: We consider Problem 1 wherein the function $f(x)$ is an unknown function. Using equations (5) and (24), we have

$$w(x,\tau)=\varphi(x)-e^{-\tau}u_1(x)-(g_1(\tau)-g_0(\tau))x-g_0(\tau), \quad 0<x<L. \quad (26)$$

The function f(x) may be approximated in terms of the shifted SKCPs as follows

$$f(x)\simeq\sum_{j=0}^{N}f_jU_{L,j}(x). \quad (27)$$

Substituting (27) into equation (25) yields

$$w(x,t)=2\sum_{k=1}^{\infty}\left(\int_0^t\int_0^L\left((\sum_{j=0}^{N}f_jU_{L,j}(\xi))g(\eta)+F(\xi,\eta)\right)\sin(k\pi\xi)e^{-(k\pi)^2(t-\eta)}d\xi d\eta\right)\sin(k\pi x). \quad (28)$$

Collocating equation (28) in N+1 nodes $(x_i,\tau),\ \ i=0,1,\dots,N$, using equation (26) and setting $w_i=w(x_i,\tau)$, we have

$$w_i=2\sum_{k=1}^{\infty}\left(\int_0^\tau\int_0^L\left((\sum_{j=0}^{N}f_jU_{L,j}(\xi))g(\eta)+F(\xi,\eta)\right)\sin(k\pi\xi)e^{-(k\pi)^2(\tau-\eta)}d\xi d\eta\right)\sin(k\pi x_i), \quad (29)$$

where

$$x_i=\frac{i+1}{N+2}L, \quad i=0,1,\dots,N.$$

Finally, considering $X=(f_i),\ W=(w_i),\ D=(d_i),\ A=(a_{ij}),\ i,j=0,1,\dots,N$ and $B=W-D$ also using equation (29), we obtain the following system

$$AX=B, \quad (30)$$

where

$$a_{ij}=2\sum_{k=1}^{\infty}\left(\int_0^\tau\int_0^L U_{L,j}(\xi)g(\eta)\sin(k\pi\xi)e^{-(k\pi)^2(\tau-\eta)}d\xi d\eta\right)\sin(k\pi x_i),$$

$$d_i=2\sum_{k=1}^{\infty}\left(\int_0^\tau\int_0^L F(\xi,\eta)\sin(k\pi\xi)e^{-(k\pi)^2(\tau-\eta)}d\xi d\eta\right)\sin(k\pi x_i).$$

Equation (30) forms a system of (N+1) equations with (N+1) unknown coefficients f_j, $j = 0,1,\ldots,N$ and the unknowns can be computed using direct methods. After determining the coefficients f_j, $j = 0,1,\ldots,N$, we find an approximation of the function $s(x,t)$ as

$$s(x,t) \simeq (\sum_{j=0}^{N} f_j U_{L,j}(x))g(t).$$

Problem 2: In this problem, we approximate the unknown function $g(t)$ in terms of the shifted SKCPs as follows

$$g(t) \simeq \sum_{j=0}^{N} b_j U_{\tau,j}(t). \tag{31}$$

Using equations (25) and (31), we obtain

$$w(x,t) = 2\sum_{k=1}^{\infty}\left(\int_0^t \int_0^L \left(f(\xi)(\sum_{j=0}^{N} b_j U_{\tau,j}(\eta)) + F(\xi,\eta)\right)\sin(k\pi\xi)e^{-(k\pi)^2(t-\eta)}d\xi d\eta\right)\sin(k\pi x). \tag{32}$$

Collocating equation (32) in N+1 nodes (x_0,t_i), $i = 0,1,\ldots,N$, then using equations (5) and (24) and finally setting $w'_i = w(x_0,t_i)$, we get

$$w'_i = 2\sum_{k=1}^{\infty}\left(\int_0^{t_i} \int_0^L \left(f(\xi)(\sum_{j=0}^{N} b_j U_{\tau,j}(\eta)) + F(\xi,\eta)\right)\sin(k\pi\xi)e^{-(k\pi)^2(t_i-\eta)}d\xi d\eta\right)\sin(k\pi x_0), \tag{33}$$

where

$$t_i = \frac{i+1}{N+2}\tau, \qquad i = 0,1,\ldots,N.$$

Now, suppose that $X' = (b_i)$, $W' = (w'_i)$, $D = (d'_i)$, $A = (a'_{ij})$, $i = 0,1,\ldots,N$, $i = j = 0,1,\ldots,N$ and $B' = W' - D'$ where

$$a'_{ij} = 2\sum_{k=1}^{\infty}\left(\int_0^{t_i} \int_0^L f(\xi)U_{\tau,j}(\eta)\sin(k\pi\xi)e^{-(k\pi)^2(t_i-\eta)}d\xi d\eta\right)\sin(k\pi x_0),$$

$$d'_i = 2\sum_{k=1}^{\infty}\left(\int_0^{t_i} \int_0^L F(\xi,\eta)\sin(k\pi\xi)e^{-(k\pi)^2(t_i-\eta)}d\xi d\eta\right)\sin(k\pi x_0).$$

Then, we have the following system of equations

$$A'X' = B', \tag{34}$$

which forms a system of $N+1$ linear equations and can be solved using direct methods to obtain the unknown coefficients b_j, $j = 0,1,\ldots,N$. Finally, the unknown source function is approximated by

$$s(x,t)=f(x)(\sum_{j=0}^{N} b_j U_{\tau,j}(t)).$$

In two above problems, the elements of vector B in the linear system (30) and B' in the system (34) come from the overspecified conditions (5) and (6). These conditions are obtained from practical measurements that are inherently contaminated with random noise. On the other hand, due to the ill-posedness of these inverse problems, (30) and (34) are ill-conditioned. Thus, the numerical reconstruction of solutions are very difficult. Hence, some special regularization methods are required to obtain an accurate approximation for solution of these systems. For this purpose, we employ the Tikhonov regularization method. By this technique, we have a minimization problem [31] as

$$\min_{X\in R^{(N+1)}} \| AX - B\|^2 + \alpha \| X\|^2, \tag{35}$$

or

$$\min_{X'\in R^{(N+1)}} \| A'X' - B'\|^2 + \alpha \| X'\|^2, \tag{36}$$

the constant $\alpha > 0$ is regularization parameter, which controls the trade-off between fidelity to the data and smoothness of the solution. Different methods have been applied for determining regularization parameter [31, 32]. The method which we apply, is the generalized cross-validation (GCV) method [33]. In our computations, we will use the Matlab codes developed by Hansen [34] for solving the ill-conditioned systems (30) and (34).

Numerical solution

In this section, we use the results obtained in the previous section to solve the main problem.

Substituting the obtained source function $s(x,t)$ into equation (1), we get the following equation

$$u_t(x,t) - u_{xx}(x,t) = s(x,t), \tag{37}$$

with conditions (2) and (3). Integrating both sides of (37) with respect to t and using initial condition (2), we have

$$u(x,t) - u_0(x,t) - \int_0^t u_{xx}(x,t')dt' = \int_0^t s(x,t')dt', \tag{38}$$

where

$$u_0(x,t) = u_0(x).$$

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We approximate the functions $u(x,t)$, $u_0(x,t)$ and $s(x,t)$ in (38) using the method mentioned in Section 2 as

$$u(x,t) \simeq \sum_{i=0}^{M}\sum_{j=0}^{M} c_{ij}\phi_{ij}^{L,\tau}(x,t) = C^T \Phi_{L,\tau}(x,t), \tag{39}$$

$$u_0(x,t) \simeq \sum_{i=0}^{M}\sum_{j=0}^{M} c'_{ij}\phi_{ij}^{L,\tau}(x,t) = C_0^T \Phi_{L,\tau}(x,t), \tag{40}$$

$$s(x,t) \simeq \sum_{i=0}^{M}\sum_{j=0}^{M} s_{ij}\phi_{ij}^{L,\tau}(x,t) = S^T \Phi_{L,\tau}(x,t), \tag{41}$$

where $M \geq 2$ and the vector C in equation (39) is unknown. Substituting approximations (39)-(41) into equation (38) and using equations (18) and (21) yield

$$UC = G, \tag{42}$$

where

$$G = C_0 + P^T S,$$

and

$$U = (I - D^2 P)^T,$$

here I is the identity matrix of order $(M+1)^2$.

Now, to solve the main problem we need to apply the boundary conditions. To this aim, the boundary conditions (3) are written using equations (22) and (23) as

$$C^T W_0^T \psi_\tau(t) = g_0(t), \tag{43}$$

$$C^T W_1^T \psi_\tau(t) = g_1(t). \tag{44}$$

The functions $g_0(t)$ and $g_1(t)$ can be approximated using equation (13) as follows

$$g_0(t) = G_0^T \psi_\tau(t) = \psi_\tau^T(t) G_0, \tag{45}$$

$$g_1(t) = G_1^T \psi_\tau(t) = \psi_\tau^T(t) G_1. \tag{46}$$

Substituting (45) and (46) into equations (43) and (44) respectively, we have

$$W_0 C = G_0$$

$$W_1 C = G_1$$

To obtain the solution of problems (1)-(3), we replace 2(M+1) rows of the augmented matrix [U;G] with the rows of the augmented matrices $[W_0;G_0]$ and $[W_1;G_1]$. In this way, the unknown vector C is determined by solving the new matrix equation.

Numerical examples

In this section, some examples are given to show the applicability and accuracy of our method. In order to demonstrate the error of the method to find the unknown source function, we introduce the notations:

$$e_N(x) = |f(x) - f_N(x)|,$$
$$e_N(t) = |g(t) - g_N(t)|$$

where $f(x)$, $f_N(x)$, $g(t)$, and $g_N(t)$ are the exact unknown source function in problem 1, its approximation using the presented method, the exact unknown source function in Problem 2 and its approximation using the presented method, respectively. The absolute values of the error are reported at some selected grid points for the examples. Also, we will use the following notation to show the absolute error for approximate solution

$$E_M(x,t) = |u(x,t) - u_M(x,t)|,$$

in which $u(x,t)$ and $u_M(x,t)$ are the exact and approximate solutions, respectively.

For noisy data, a randomly distributed perturbation $\delta \times randn$ is added to the additional data, in the form

$$\varphi^{\delta}(x_i) = \varphi(x_i) + \delta \times randn,$$

and

$$\chi^{\delta}(t_i) = \chi(t_i) + \delta \times randn,$$

for $i = 0,1,\dots,N$, where *randn* is a normal random distribution with zero mean and unit standard deviation generated by MATLAB function *randn* and δ indicates the level of noise.

The computations were performed on a personal computer using a 2.20 GHz processor and the codes were written in *Mathematica* 10. In all the examples of this section we have used k=1 to 20 in order to calculate the matrices A and B in equation (5) and A' and B' in equation (6).

Example 5.1. As the first example, consider the following inverse problem

$$u_t - u_{xx} = (1+\pi^2) sin(\pi x) g(t), \quad 0 < x < 1, 0 < t \le 3,$$
$$u(x,0) = sin(\pi x),$$
$$u(0,t) = u(1,t) = 0,$$

with overspecified condition as

$$u(0.5,t) = e^t, \quad 0 < t \le 3,$$

which has the unique solution given by

$$u(x,t) = e^{t} sin(\pi x),$$
$$g(t) = e^{t}.$$

We have applied the presented method to solve this problem. Numerical results are displayed in Figures 1-4 and Table 1. In Figure 1 and Figure 2 the numerical results for obtaining the unknown source function $g(t)$ with $N = 10$ and their absolute errors for various noise levels are plotted. Table 1 displays the absolute error of the source function at some selected points with N=4, 6, 8, 10. Also, Figure 3 and Figure 4 show the absolute error of approximate solutions obtained with M=10 at $x = 0.5$ and $t = 2.5$, respectively. Note that to obtain the numerical results for the unknown function $u(x,t)$, we have used the function $g(t)$ obtained by N=10.

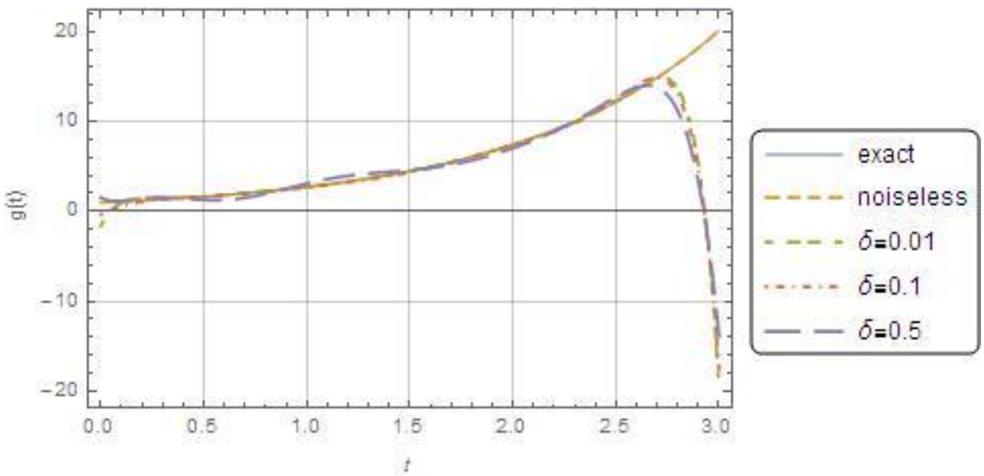

Figure 1. Plot of the function g(t) and its approximations with N=10 for Example 5.1 with the various noise levels.

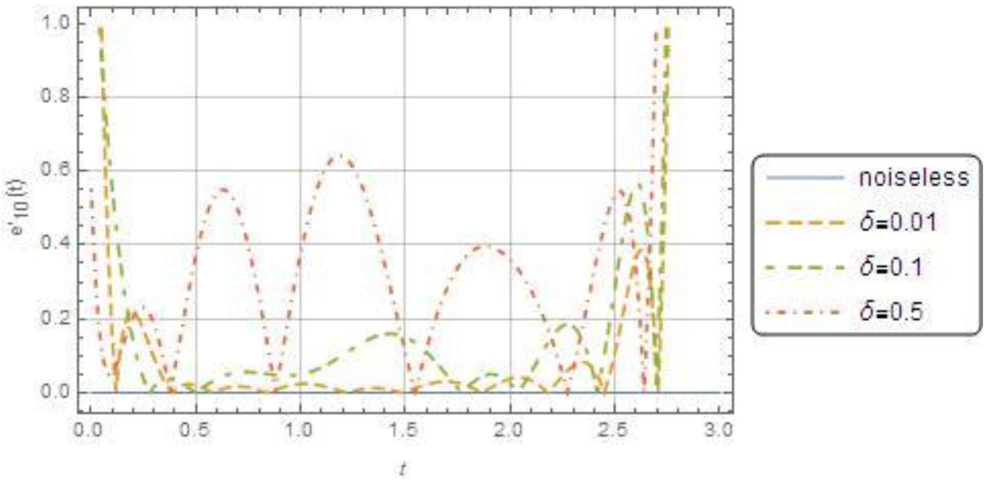

Figure 2. Plot of the function $e'_{10}(t)$ in Example 5.1 with the various noise levels.

Table 1. Noiseless numerical results for the absolute error of the source function at some selected points for Example 5.1.

t	N=4	N=6	N=8	N=10
0.0	6.04×10^{-2}	1.54×10^{-3}	2.21×10^{-5}	2.07×10^{-7}
0.3	4.24×10^{-3}	8.74×10^{-5}	1.49×10^{-6}	7.61×10^{-9}
0.6	4.20×10^{-3}	1.41×10^{-5}	2.62×10^{-7}	7.24×10^{-10}
0.9	7.93×10^{-5}	1.75×10^{-5}	8.62×10^{-8}	5.73×10^{-11}
1.2	1.61×10^{-3}	1.30×10^{-5}	4.68×10^{-8}	6.24×10^{-11}
1.5	8.72×10^{-4}	8.02×10^{-6}	3.98×10^{-8}	1.23×10^{-10}
1.8	1.44×10^{-3}	1.93×10^{-7}	5.24×10^{-8}	2.31×10^{-10}
2.1	3.41×10^{-3}	2.10×10^{-5}	1.09×10^{-7}	5.44×10^{-10}
2.4	1.93×10^{-3}	1.07×10^{-4}	4.01×10^{-7}	1.68×10^{-9}
2.7	5.03×10^{-2}	6.72×10^{-4}	3.33×10^{-6}	3.21×10^{-9}
3.0	2.53×10^{-1}	8.99×10^{-3}	1.82×10^{-4}	2.36×10^{-6}

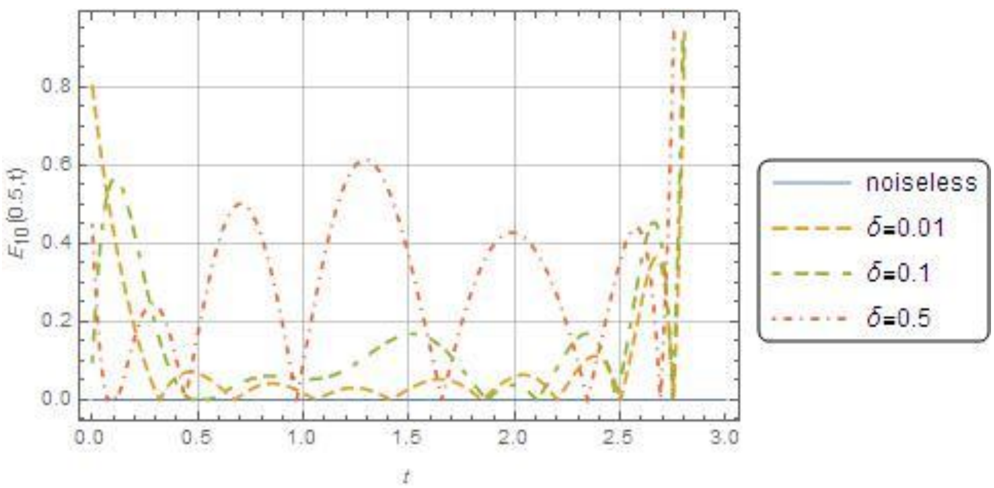

Figure 3. The absolute errors of approximate solutions for Example 5.1 when $x = 0.5$.

Example 5.2. Consider Problem 1 with L=1, $\tau = 1$ and

$$u(x,0) = e^{2x},$$
$$u(0,t) = e^{t},\ u(1,t) = e^{t+2},$$
$$u(x,1) = e^{2x+1},\ g(t) = e^{t}.$$

The exact solution of this problem is

$$u(x,t) = e^{2x+t},$$
$$f(x) = -3e^{2x}.$$

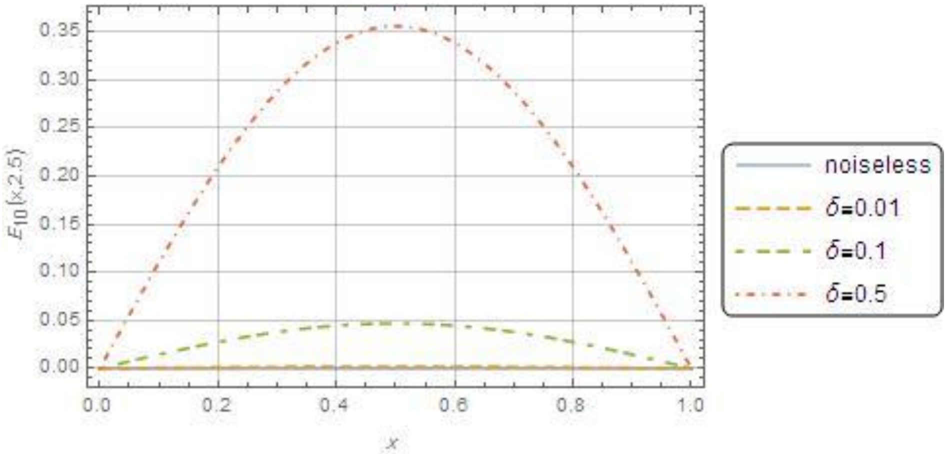

Figure 4. The absolute errors of approximate solutions for Example 5.1 when $t = 2.5$.

Figures 5-8 and Table 2 report the numerical results for this example. Figures 5 and 6 show the numerical results for $f(x)$ with $N = 6$ by various noise levels. After substituting the approximate $f(x)$ obtained with $N = 6$ into the main problem, the numerical approximations of $u(x,t)$ with $M = 6$ have been produced so that the numerical results at x=0.5 and $t = 1$ are given in Figure 7 and Figure 8, respectively. Table 2, displays the absolute error at some selected grid points with different values of M after substituting the noiseless result obtained by $N = 6$.

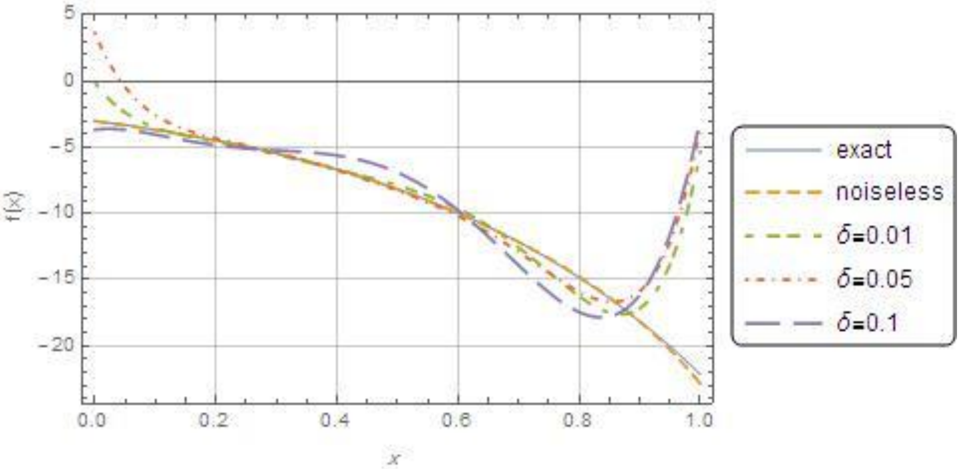

Figure 5. Plot of the function f(x) and its approximations with N=6 in Example 5.2 with the various noise levels.

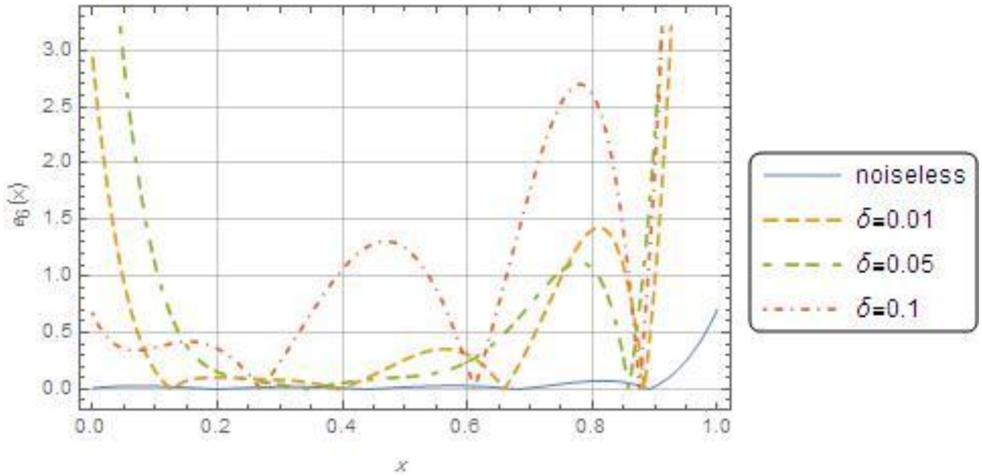

Figure 6. Plot of the function $e_6(x)$ in Example 5.2 with the various noise levels.

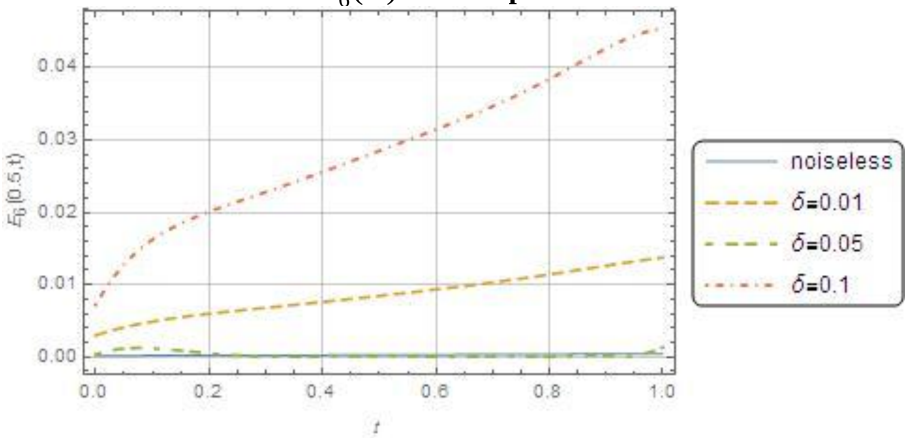

Figure 7. The absolute errors of approximate solutions for Example 5.2 when $\mathbf{x = 0.5}$.

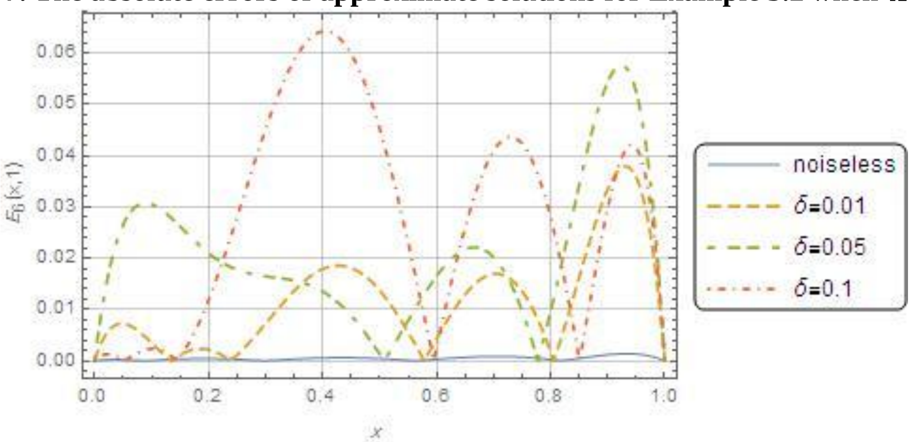

Figure 8. The absolute errors of approximate solutions for Example 5.2 when t=1.

Table 2. Numerical results for the absolute error at some selected grid points when t=1 for Example 5.2.

x	M=2	M=4	M=6
0.1	3.82×10^{-1}	9.84×10^{-4}	1.79×10^{-4}
0.2	5.48×10^{-1}	5.74×10^{-3}	2.68×10^{-5}
0.3	5.47×10^{-1}	9.99×10^{-3}	2.86×10^{-4}
0.4	4.14×10^{-1}	7.93×10^{-3}	6.63×10^{-4}
0.5	1.94×10^{-1}	5.83×10^{-4}	6.19×10^{-4}
0.6	5.94×10^{-2}	7.99×10^{-3}	2.42×10^{-4}
0.7	2.81×10^{-1}	1.28×10^{-2}	1.56×10^{-4}
0.8	3.91×10^{-1}	1.05×10^{-2}	9.08×10^{-4}
0.9	2.90×10^{-1}	3.00×10^{-3}	1.90×10^{-3}

Example 5.3. In this example, we compare our method with some previous approaches for estimating the unknown source term. Consider problems (1)-(4) with $g(\mathrm{t})=\mathrm{e}^{-0.3t}$ and $u_0(\mathrm{x})=\mathrm{g}_0(\mathrm{t})=\mathrm{g}_1(\mathrm{t})=0.$ With these assumptions, the exact space dependent source function is $f(\mathrm{x})=\sin(\pi\mathrm{x})$. For obtaining noisy data, similar to [21], the term $2\delta(\mathrm{rand}-0.5)$ is added to $\varphi(x)=\dfrac{e^{-0.3}-e^{-\pi^2}}{\pi^2-0.3}$. The source function and its approximations for various noise levels are shown in Figure 9. Also Table 3 and Table 4 show the relative errors $\dfrac{\|f(x)-\bar{f}(x)\|}{\|f(x)\|}$ of the numerical results $\bar{f}(x)$ to estimate $f(x)$ in comparison with some other numerical methods, where $\|.\|$ is L^2 -norm on $[0,1]$.

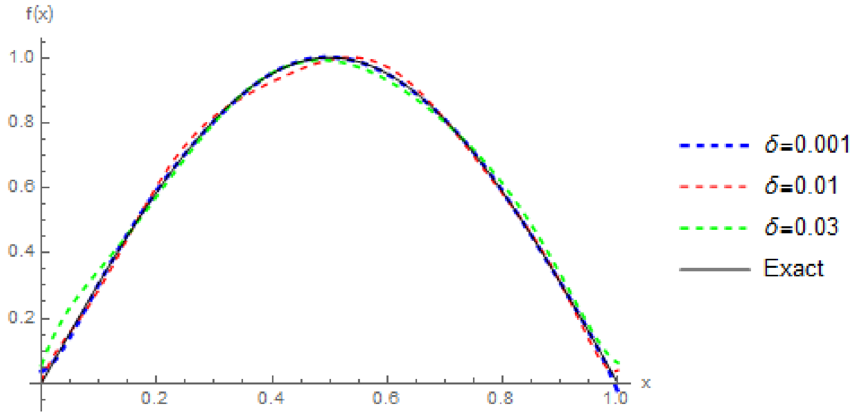

Figure 8. Exact source function (Black) and its approximations N=10 in Example 5.3 for additional data with the noise level $\delta=0.001$ (Blue), $\delta=0.01$ (Red) and $\delta=0.03$ (Green).

Table 3. Comparison between the errors of the estimated source term when N=5 with some other methods for Example 5.3.

Noise level	Sinc-Galerkin method [22]	Spectral method based on Legendre polynomial	Ritz method [21]	Our method
$\delta = 0.001$	0.0942	0.1223	0.0068	0.0096
$\delta = 0.01$	0.2644	0.1789	0.0319	0.0116
$\delta = 0.03$	0.4660	0.2037	0.1038	0.0649

Table 4. Comparison between the errors of the estimated source term when N=10 with some other methods for Example 5.3.

Noise level	Sinc-Galerkin method [22]	Spectral method based on Legendre polynomial	Ritz method [21]	Our method
$\delta = 0.001$	0.0461	0.0780	0.0095	0.0071
$\delta = 0.01$	0.2871	0.1315	0.0262	0.0192
$\delta = 0.03$	0.3735	0.1770	0.0995	0.0391

Conclusion

In this work, we applied a spectral method based on the second kind Chebyshev polynomials to the numerical solution of a parabolic inverse problem with unknown source function. First, we introduced a method to find an approximation of the unknown source function by considering this function in the form of a linear combination of Chebyshev polynomials. A system of linear equations was constructed to obtain the coefficients of this combination. Since this system of equations was ill-conditioned, the Tikhonov regularization technique was applied to find a stable solution. Then, by substituting the result into the main problem and using the operational matrices, which are all sparse matrices, we obtained an approximation of the solution. Although using Chebyshev polynomials to solve partial differential equations is a classic method, but using these polynomials to obtain the unknown source function in inverse parabolic problems, according to the numerical algorithm which is presented in this work, is new. The numerical results show that the proposed method in this paper is a reliable method and gives good results even when we use a small number of the basis functions.

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