A modified VIM for solving an inverse heat conduction problem

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Received: 18 Nov 2013 Revised: 10 Nov 2014

Abstract

In this paper, we will use a modified variational iteration method (MVIM) for solving an inverse heat conduction problem (IHCP). The approximation of the temperature and the heat flux at $x = 0$ are considered. This method is based on the use of Lagrange multipliers for the identification of optimal values of parameters in a functional in Euclidean space. Applying this technique, a rapid convergent sequence to the exact solution is produced. Moreover, this method does not require any discretization, linearization or small perturbation, thus it can be considered as an efficient method to solve this problem. To show the strength and capability of this method, some examples are given.

Keywords: IHCP, Unstable, Ill-posed, MVIM, BFM, FSM, FDM, SFDM.

Introduction

Inverse heat conduction problems have many applications in various branches of science and engineering. In remote sensing, oil exploration, nondestructive evaluation of material and determination of the earth's interior structure. One of the applications may be the determination of the surface heat flux histories of reentering heat shield [1].

Inverse problems are in nature 'unstable' because the unknown solutions and parameters have to be determined from indirect observable data which contain measurement error.

The major difficulty in establishing any numerical algorithm for approximating the solution is the ill-posedness of the problem and the ill-conditioning of the resulting discretized matrix.

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[DOR: 20.1001.1.25882546.1395.2.1.8.9]
[DOI: 10.29252/mmr.2.1.89]

A number of techniques have been proposed for solving the inverse problems, such as the boundary element method (BEM) [2], the method of fundamental solutions [3,4], genetic algorithm [5], regularization method [6] and etc [7-15]. In this study, we use a modified VIM to construct a solution to an IHCP. The VIM was first suggested by Ji-Huan He [16-23]. This method is based on the use of Lagrange multipliers for the identification of optimal values of parameters in a functional. This method constructs a rapidly convergent sequence to the exact solution. Moreover, VIM does not require any discretization, linearization or small perturbation. This method is effectively, convenience and accurate. Thus, it has been extensively applied to various kinds of linear and nonlinear problems [24-28].

This paper is organized as follows: In Section 2, description of the problem is presented. Numerical procedure is introduced in Section 3. In Section 4, some examples are given. A conclusion of paper is considered in Section 5.

Description of the problem

In this section, we consider the following IHCP, in the dimensionless form:

$$T_t(x,t) = T_{xx}(x,t) + h(x,t); \qquad 0 < x < 1, \quad 0 < t < t_{fin}, \tag{1}$$

$$T(x,0) = \psi(x); \qquad 0 \le x \le 1, \tag{2}$$

$$T(0,t) = p(t); \qquad 0 \le t \le t_{fin}, \tag{3}$$

$$T(1,t) = q(t); \qquad 0 \le t \le t_{fin}, \tag{4}$$

and the over specified condition:

$$T(x_1,t) = g(t); \qquad 0 < x_1 < 1, \quad 0 \le t \le t_{fin}, \tag{5}$$

where h is known heat source, $\varphi(x)$ is continuous known function, $q(t)$ and $g(t)$ are infinitely differentiable known functions, $x_1 \in (0,1)$ is the interior location of a thermocouple recording the temperature measurement (5) and t_{fin} represent the final time of interest for the time evolution of the problem, while the temperature $T(0,t) = p(t)$ and heat flux $T_x(0,t)$ are unknown which remain to be determined from some interior temperature measurements.

The problem (1)-(5) may be divided into two separate problems, as shown in Figure1.

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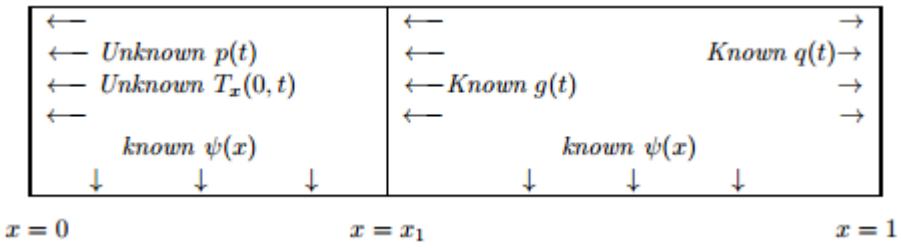

Figure 1. Inverse heat conduction problem (1)-(5)

The first problem is:

$$T_t(x,t)=T_{xx}(x,t)+h(x,t); \qquad x_1<x<1, \quad 0<t<t_{fin}, \tag{6}$$

$$T(x,0)=\psi(x); \qquad x_1\le x\le 1, \tag{7}$$

$$T(x_1,t)=g(t); \qquad 0\le t\le t_{fin}, \tag{8}$$

$$T(1,t)=q(t); \qquad 0\le t\le t_{fin}. \tag{9}$$

This problem may be analyzed as a direct problem for the portion of the body from $x=x_1$ to $x=1$ with known boundary conditions. There is a unique stable solution to the direct problem (6)-(9) and may be found in [29]. The second problem is the following IHCP:

$$T_t(x,t)=T_{xx}(x,t)+h(x,t); \qquad 0<x<x_1, \quad 0<t<t_{fin}, \tag{10}$$

$$T(x_1,t)=g(t); \qquad 0\le t\le t_{fin}, \tag{11}$$

$$T_x(x_1,t)=k(t); \qquad 0\le t\le t_{fin}. \tag{12}$$

The heat flux at $x=x_1$ $(i.e.\, T_x(x_1,t)=k(t))$ can be obtained from the solution of the direct problem (6)-(9) [30, 31]. The problem (10)-(12) is called Cauchy problem. The Cauchy problem is ill-posed. The solution of the problem (10) through (12) exists and is unique but not always stable [29].

In the next section, the above IHCP will be considered, the heat in the body and heat flux at the boundary $x=0$ will be obtained by solving this problem numerically.

Numerical procedures

Consider the general differential equation:

$$LT(x)+NT(x)=\varphi(x); \qquad a<x<b,$$

where L and N are linear and nonlinear operators, respectively, and φ is an inhomogeneous term. According to VIM, we construct a correction functional as follows:

$$T_{n+1}(x)=T_n(x)+\int_a^x \lambda(s)(LT_n+N\tilde{T}_n(s)-\varphi(s))ds,$$

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where λ is a Lagrange multiplier, which can be identified optimally via the variational theory, $\tilde{T}_n$ is a restricted variation, i.e. $\delta\tilde{T}_n = 0$ [16, 25]. Now, we need to determine the Lagrangian multiplier λ. Then by using the determined Lagrangian multiplier and an initial value $T_0(x)$ [11, 18-20, 25, 32-33], the successive approximations $T_{n+1}(x)$, $n \geq 0$, of the solution $T(x)$ will be readily obtained as follows:

$$T_{n+1}(x) = T_n(x) + \int_a^x \lambda(s)(LT_n + NT_n(s) - \varphi(s))ds.$$

Now, we solve the problem (10)-(12) with MVIM. For equation (10), the correction functional can be expressed as follows:

$$T_{n+1}(x,t) = T_n(x,t) + \int_{x_1}^x \lambda(s)(T_{n_{ss}}(s,t) - \tilde{T}_{n_t}(s,t) + h(s,t))ds,$$

where $\tilde{T}_n$ is a restricted variation and λ is the Lagrange multiplier.

To find the optimal value of λ, we have:

$$\delta T_{n+1}(x,t) = \delta T_n(x,t) + \delta\int_{x_1}^x \lambda(s)(T_{n_{ss}}(s,t) - \tilde{T}_{n_t}(s,t) + h(s,t))ds = 0.$$

After some calculation, we obtain the following stationary conditions:

$$\lambda''(s) = 0, \quad 1 - \lambda'(x) = 0, \quad \lambda(x) = 0.$$

So, we have:

$$\lambda(s) = s - x.$$

Therefore, we obtain the following iteration formula:

$$T_{n+1}(x,t) = T_n(x,t) + \int_{x_1}^x (s-x)(T_{n_{ss}}(s,t) - T_{n_t}(s,t) + h(s,t))ds, \tag{13}$$

where T_0 may be selected as any function that just satisfies, at least, the initial or boundary conditions [18, 32-36] but according to Adomian's decomposition method (ADM) in x direction which is equivalent to the VIM in $x-$ direction [38], we assume $LT_0(x,t) = 0$ or $T_0 = B + (x - x_1)C$, where B and C can be determined from the boundary conditions, for simplicity, as the initial approximation [18].

So, taking $T_0(x,t) = g(t) + (x - x_1)k(t)$ as an initial value, we can find the n-order approximate solution $T_n(x,t)$ of (10).

For the convergence of the sequence obtained via the MVIM and its rate, we recall Banach's fixed point theorem:

Theorem: [37] Let X be a Banach space and:

$$A: X \to X,$$

is a nonlinear map, and suppose that:

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$$\left\|A[T]-A[\bar{T}]\right\| \le \gamma \left\|T-\bar{T}\right\|; \qquad T,\bar{T} \in X, \tag{14}$$

for some constant $\gamma < 1$. Then A has a unique fixed point. Furthermore, the sequence:

$$T_{n+1} = A[T_n], \tag{15}$$

with an arbitrary choice of $T_0 \in X$, converges to the fixed point of A and:

$$\left\|T_k - T_l\right\| \le \left\|T_1 - T_0\right\| \sum_{j=l-1}^{k-2} \gamma^j. \tag{16}$$

According to the above theorem, for the linear map:

$$A[T] = T(x,t) + \int_{x_1}^{x} \lambda(s)(T_{ss}(s,t) - T_t(s,t) + h(s,t))ds, \tag{17}$$

a sufficient condition for the convergence of MVIM is strictly contraction of A. Furthermore, sequence (15) converges to the foxed point of A, which is also the solution of the linear differential equation (10). In the above theorem, the rate of convergence depends on γ and therefore, in the MVIM, the rate of convergence depends on λ.

Numerical results and discussion

In this section, we are going to demonstrate some numerical results for the temperature $T(0,t) = p(t)$ and heat flux $T_x(0,t)$ in the inverse problem (1)-(5). All the computations are performed on the PC (pentium(R) 4 CPU 3.20 GHz).

Example 1. Let us consider the following one-dimensional inverse parabolic problem [3]:

$$T_t(x,t) = T_{xx}(x,t); \qquad 0 < x < 1, \quad 0 < t < 1,$$
$$T(x,0) = \cos(x); \qquad 0 \le x \le 1,$$
$$T(0,t) = p(t); \qquad 0 \le t \le 1,$$
$$T(1,t) = \exp(-t)\cos(1); \qquad 0 \le t \le 1,$$
$$T(0.2,t) = \exp(-t)\cos(0.2); \qquad 0 \le t \le 1.$$

The exact solution of this problem is:

$$T(x,t) = \exp(-t)\cos(x), \qquad T_x(x,t) = -\exp(-t)\sin(x).$$

We begin with the initial approximation

$$T_0 = \exp(-t)\cos(0.2) - (x-0.2)\exp(-t)\sin(0.2).$$

Then we obtain:

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$$T_1 = T_0 + \int_{0.2}^{x} (s-x)(T_{0_{ss}} - T_{0_t})ds$$

$$= T_0 + \exp(-t)[-\frac{(x-0.2)^2}{2!}\cos(0.2) - \frac{(x-0.2)^3}{3!}\sin(0.2)]$$

$$= \exp(-t)[(1-\frac{(x-0.2)^2}{2!})\cos(0.2) - ((x-0.2) - \frac{(x-0.2)^3}{3!})\sin(0.2)]$$

In the same way, we compute:

$$T_2 = \exp(-t)[(1-\frac{(x-0.2)^2}{2!} + \frac{(x-0.2)^4}{4!})\cos(0.2)$$

$$-((x-0.2) - \frac{(x-0.2)^3}{3!} + \frac{(x-0.2)^5}{5!})\sin(0.2)],$$

$$\vdots$$

$$T_n = \exp(-t)[(1-\frac{(x-0.2)^2}{2!} + \cdots + (-1)^n \frac{(x-0.2)^{2n}}{(2n)!})\cos(0.2)$$

$$-((x-0.2) - \frac{(x-0.2)^3}{3!} + \cdots + (-1)^{n+1} \frac{(x-0.2)^{2n-1}}{(2n-1)!})\sin(0.2)].$$

According to the Theorem for linear map A, a sufficient condition for convergence of the MVIM is strictly contraction of A. Therefore, we have

$$\| T_0 - T \| = \| \exp(-t)(\cos(0.2) - (x-0.2)\sin(0.2) - \cos(x)) \|,$$

$$\| T_1 - T \| = \left\| \exp(-t)[(1-\frac{(x-0.2)^2}{2!})\cos(0.2) - ((x-0.2) - \frac{(x-0.2)^3}{3!})\sin(0.2) - \cos(x)] \right\|$$

$$\leq \| T_0 - T \| \left\| 1 - \frac{T_0 - T_1}{T_0 - T} \right\|.$$

Since, for all $x \in [0, 0.2-\delta) \cup (0.2+\delta, 1]$, $\delta = 0.01$, we have

$$\left\| 1 - \frac{T_0 - T_1}{T_0 - T} \right\| \leq \gamma = 0.0564 < 1,$$

therefore,

$$\| T_1 - T \| \leq \gamma \| T_0 - T \|,$$

$$\| T_2 - T \| = \| \exp(-t)[(1-\frac{(x-0.2)^2}{2!} + \frac{(x-0.2)^4}{4!})\cos(0.2)$$

$$-((x-0.2) - \frac{(x-0.2)^3}{3!} + \frac{(x-0.2)^5}{5!})\sin(0.2) - \cos(x)] \|$$

$$\leq \| T_1 - T \| \left\| 1 - \frac{T_1 - T_2}{T_1 - T} \right\|,$$

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But, $\forall x \in [0, 0.2-\delta) \bigcup (0.2+\delta, 1],\ \delta = 0.01,$

$$\left\| 1 - \frac{T_1 - T_2}{T_1 - T} \right\| \le 0.0218 < \gamma,$$

Thus, $\| T_2 - T \| \le \gamma^2 \| T_0 - T \|$.

$$\| T_3 - T \| = \| \exp(-t)[(1 - \frac{(x-0.2)^2}{2!} + \frac{(x-0.2)^4}{4!} - \frac{(x-0.2)^6}{6!}) \cos(0.2)$$

$$- ((x-0.2) - \frac{(x-0.2)^3}{3!} + \frac{(x-0.2)^5}{5!} - \frac{(x-0.2)^7}{7!}) \sin(0.2) - \cos(x)] \|$$

$$\le \| T_2 - T \| \left\| 1 - \frac{T_2 - T_3}{T_2 - T} \right\|.$$

Since, for all $x \in [0, 0.2-\delta) \bigcup (0.2+\delta, 1],\ \delta = 0.01,$ we have

$$\left\| 1 - \frac{T_2 - T_3}{T_2 - T} \right\| \le 0.0115 < \gamma,$$

thus,

$$\| T_3 - T \| \le \gamma^3 \| T_0 - T \|,$$

$$\vdots$$

$$\| T_n - T \| \le \gamma^n \| T_0 - T \|.$$

Therefore, $\lim_{n\to\infty} \| T_n - T \| \le \lim_{n\to\infty} \gamma^n \| T_0 - T \| = 0,$ that is

$$T(x,t) = \lim_{n\to\infty} T_n(x,t) = \exp(-t)\cos(x),$$

which is the exact solution.

Tables 1 and 2 and Figures 1 and 2 show the comparison between exact and approximate solutions of $T(0,t)$ and $T_x(0,t)$ resulted from MVIM, base function method [3] (BFM) , fundamental solution method presented in [3, 4] (FSM), finite difference method (FDM) and semi finite difference method (SFDM) developed in [3, 39, 40]. In comparison with the methods in Refs. 3, 4, 39 and 40, the numerical results show that the MVIM is more accurate. In all tables n presents the iteration number in MVIM.

Table1. The comparison between exact and MVIM, BFM [3], FSM [3, 4], FDM [3,40] and SFDM [3, 41] solutions for $T(0,t)$.

t	Exact	MVIM	MVIM	BFM[3]	FSM[3,4]	FDM[3,40]	SFDM[3,41]
-	-	n=2	n=4	-	-	-	-
0.1	0.904837418	0.904837497	0.904837418	0.904837418	0.904839370	0.9033859	0.9228582
0.3	0.740818220	0.740818285	0.740818220	0.740818220	0.740818689	0.7396719	0.7555724
0.5	0.606530659	0.606530712	0.606530659	0.606530659	0.606530801	0.6055922	0.6186104
0.7	0.496585303	0.496585347	0.496585303	0.496585303	0.496585292	0.4958169	0.5064753
0.9	0.406569659	0.406569695	0.406569659	0.406569659	0.406569906	0.4059406	0.4146669

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Table2. The comparison between exact and MVIM, BFM [3], FSM [4], FDM [3, 40] and SFDM [3, 41] solutions for $T_x(0,t)$.

t	Exact	MVIM	MVIM	BFM[3]	FSM[3,4]	FDM[3,40]	SFDM[3,41]
-	-	n=2	n=4	-	-	-	-
0.1	0	-2.37E-06	-1.25E-12	2.19 E-10	-9.31E-06	7.56E-03	-1.64E-02
0.3	0	-1.94E-06	-1.02E-12	1.79E-10	-2.11E-06	5.78E-3	-1.34E-02
0.5	0	-1.59E-06	-8.41E-13	1.47E-10	-1.03E-06	4.73E-3	-1.10E-02
0.7	0	-1.30E-06	-6.89E-13	1.20E-10	+2.41E-07	3.87E-10	- 9.02E-10
0.9	0	-1.06E-06	-5.64E-13	9.86E-11	-1.09E-06	3.17E-03	-7.38E-03

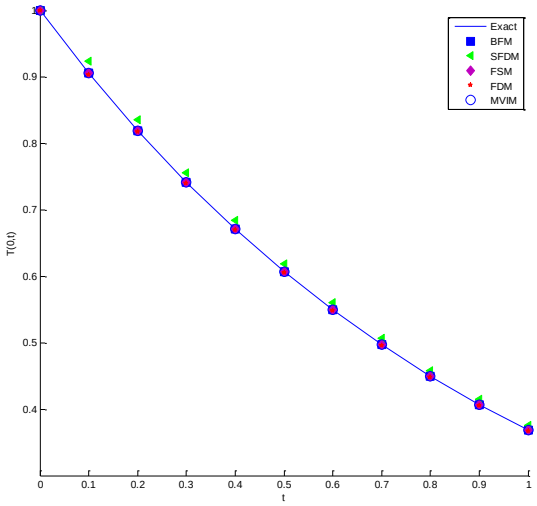

Figure2. The comparison between exact and MVIM, BFM [3], FSM [3, 4], FDM [3,40] and SFDM [3, 41] solutions for $T(0,t)$

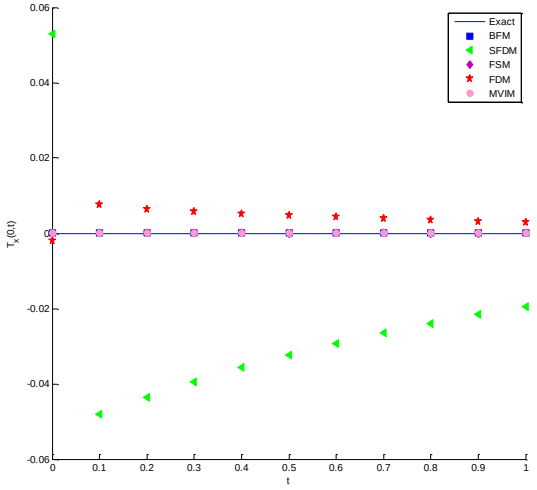

Figure3. The comparison between exact and MVIM, BFM [3], FSM [4], FDM [3, 40] and SFDM [3, 41] solutions for $T_x(0,t)$

Example2. Now, consider the problem [3]:

$$T_t(x,t) = T_{xx}(x,t); \qquad 0 < x < 1, \quad 0 < t < 1,$$

$$T(x,0) = 2(\sin(2x) + \cos(2x)) + \frac{1}{4}x^4; \qquad 0 \le x \le 1,$$

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$$T(0,t)=p(t);\qquad 0\le t\le 1,$$

$$T(1,t)=2\exp(-4t)(\sin(2)+\cos(2))+3(t^2+t+\frac{1}{12});\qquad 0\le t\le 1,$$

$$T(0.2,t)=2\exp(-4t)(\sin(0.4)+\cos(0.4))+3(t^2+0.04t+\frac{0.0016}{12});\qquad 0\le t\le 1.$$

The exact solution of this problem is:

$$T(x,t)=2\exp(-4t)(\sin(2x)+\cos(2x))+3(t^2+tx^2+\frac{1}{12}x^4),$$

$$T_x(x,t)=2\exp(-4t)(2\cos(2x)-2\sin(2x))+3(2tx+\frac{1}{3}x^3).$$

We find $T_x(0.2,t)=2\exp(-4t)(2\cos(0.4)-2\sin(0.4))+3(0.4t+\frac{0.008}{3}).$

Now beginning with $T_0=T(0.2)+(x-0.2)T_x(0.2,t),$ we obtain:

$$\begin{aligned}T_1&=T_0+\int_{0.2}^{x}(s-x)(T_{0_{ss}}-T_{0_t})ds\\&=2\exp(-4t)[(\sin(0.4)+\cos(0.4))(1-\frac{(2x-0.4)^2}{2!})\\&+(\cos(0.4)-\sin(0.4))((2x-0.4)-\frac{(2x-0.4)^3}{3!})]\\&+3t^2+3tx^2+0.2x^3-0.06x^2+0.008x-0.0004.\end{aligned}$$

In the same way, we compute:

$$\begin{aligned}T_2&=2\exp(-4t)[(\sin(0.4)+c\,\mathrm{os}(0.4))(1-\frac{(2x-0.4)^2}{2!}+\frac{(2x-0.4)^4}{4!})\\&+(\cos(0.4)-\sin(0.4))((2x-0.4)-\frac{(2x-0.4)^3}{3!}+\frac{(2x-0.4)^5}{5!})]\\&+3(t^2+tx^2+\frac{1}{12}x^4),\\&\vdots\end{aligned}$$

$$\begin{aligned}T_n&=2\exp(-4t)[(\sin(0.4)+c\,\mathrm{os}(0.4))(1-\frac{(2x-0.4)^2}{2!}+\cdots+(-1)^n\frac{(2x-0.4)^{2n}}{(2n)!})\\&+(\cos(0.4)-\sin(0.4))((2x-0.4)-\frac{(2x-0.4)^3}{3!}+\cdots+(-1)^{n+1}\frac{(2x-0.4)^{2n-1}}{(2n-1)!})]\\&+3(t^2+tx^2+\frac{1}{12}x^4).\end{aligned}$$

To show the convergence of A, consider $T=u+v$ where

$u=2\exp(-4t)(\sin(2x)+\cos(2x))$ and $v=3(t^2+tx^2+\frac{1}{12}x^4)$. Then

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$$\| T_0 - T \| = \| u_0 + v_0 - u - v \|$$

$$= \| 2\exp(-4t)\left[(\sin 0.4 + \cos 0.4) + (2x - 0.4)(\cos 0.4 - \sin 0.4) - (\sin 2x + \cos 2x)\right]$$

$$+ 3(t^2 + 0.04t + \frac{0.0016}{12}) + 3(x - 0.2)(0.4t + \frac{0.008}{3}) - 3(t^2 + tx^2 + \frac{1}{12}x^4) \|$$

$$\leq \left\| 2\exp(-4t)\left[(\sin 0.4 + \cos 0.4) + (2x - 0.4)(\cos 0.4 - \sin 0.4) - (\sin 2x + \cos 2x)\right] \right\|$$

$$+ \left\| 3(t^2 + 0.04t + \frac{0.0016}{12}) + 3(x - 0.2)(0.4t + \frac{0.008}{3}) - 3(t^2 + tx^2 + \frac{1}{12}x^4) \right\|$$

$$= \| u_0 - u \| + \| v_0 - v \|,$$

$$\| T_1 - T \| = \| u_1 + v_1 - u - v \|$$

$$= \| 2\exp(-4t)[(\sin 0.4 + \cos 0.4)(1 - \frac{(2x - 0.4)^2}{2!})$$

$$+ (\cos 0.4 - \sin 0.4)((2x - 0.4) - \frac{(2x - 0.4)^3}{3!}) - (\sin 2x + \cos 2x)]$$

$$+ 3(t^2 + 0.04t + \frac{0.0016}{12}) + 3(x - 0.2)(0.4t + \frac{0.008}{3}) - 3(t^2 + tx^2 + \frac{1}{12}x^4) \|$$

$$\leq \| 2\exp(-4t)[(\sin 0.4 + \cos 0.4)(1 - \frac{(2x - 0.4)^2}{2!})$$

$$+ (\cos 0.4 - \sin 0.4)((2x - 0.4) - \frac{(2x - 0.4)^3}{3!}) - (\sin 2x + \cos 2x)] \|$$

$$+ \| 3t^2 + 3tx^2 + 0.2x^3 - 0.06x^2 + 0.008x - 0.0004 - 3(t^2 + tx^2 + \frac{1}{12}x^4) \|$$

$$= \|u_1 - u\| + \|v_1 - v\|$$

$$\leq \|u_0 - u\| \left\| 1 - \frac{u_0 - u_1}{u_0 - u} \right\| + \| v_0 - v \| \left\| 1 - \frac{v_0 - v_1}{v_0 - v} \right\|.$$

Since, for all $x \in [0, 0.2 - \delta) \bigcup (0.2 + \delta, 1]$, $\delta = 0.011$, we have:

$$\left\| 1 - \frac{u_0 - u_1}{u_0 - u} \right\| \leq \gamma_1 = 0.2233 < 1, \qquad \left\| 1 - \frac{v_0 - v_1}{v_0 - v} \right\| \leq \gamma_2 = 0.4211 < 1,$$

therefore,

$$\| T_1 - T \| \leq \gamma \left(\|u_0 - u\| + \|v_0 - v\| \right),$$

where $\gamma = \max\{\gamma_1, \gamma_2\}$.

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$$\| T_2 - T \| = \| 2\exp(-4t)[(\sin 0.4 + \cos 0.4)(1 - \frac{(2x-0.4)^2}{2!} + \frac{(2x-0.4)^4}{4!})$$

$$+ (\cos 0.4 - \sin 0.4)((2x-0.4) - \frac{(2x-0.4)^3}{3!} + \frac{(2x-0.4)^5}{5!}) - (\sin 2x + \cos 2x)] \|$$

$$\le \|u_1 - u\| \left\| 1 - \frac{u_1 - u_2}{u_1 - u} \right\|.$$

But, $\forall x \in [0, 0.2-\delta) \bigcup (0.2+\delta, 1]$, $\delta = 0.011$,

$$\left\| 1 - \frac{u_1 - u_2}{u_1 - u} \right\| \le 0.0857 < \gamma.$$

Thus, $\| T_2 - T \| \le \gamma^2 \left(\|u_0 - u\| + \|v_0 - v\| \right)$.

$$\| T_3 - T \| = \| 2\exp(-4t)[(\sin 0.4 + \cos 0.4)(1 - \frac{(2x-0.4)^2}{2!} + \frac{(2x-0.4)^4}{4!} - \frac{(2x-0.4)^6}{6!})$$

$$+ (\cos 0.4 - \sin 0.4)((2x-0.4) - \frac{(2x-0.4)^3}{3!} + \frac{(2x-0.4)^5}{5!} - \frac{(2x-0.4)^7}{7!})$$

$$- (\sin 2x + \cos 2x)] \|$$

$$\le \|T_2 - T\| \left\| 1 - \frac{T_2 - T_3}{T_2 - T} \right\|.$$

Since, for all $x \in [0, 0.2-\delta) \bigcup (0.2+\delta, 1]$, $\delta = 0.011$, we have

$$\left\| 1 - \frac{T_2 - T_3}{T_2 - T} \right\| \le 0.0456 < \gamma,$$

thus,

$$\| T_3 - T \| \le \gamma^3 \left(\|u_0 - u\| + \|v_0 - v\| \right),$$

$$\vdots$$

$$\| T_n - T \| \le \gamma^n \left(\|u_0 - u\| + \|v_0 - v\| \right).$$

Therefore, $\lim_{n\to\infty} \| T_n - T \| \le \lim_{n\to\infty} \gamma^n \left(\|u_0 - u\| + \|v_0 - v\| \right) = 0$, that is

$$T(x,t) = \lim_{n\to\infty} T_n(x,t) = 2\exp(-4t)(\sin(2x) + \cos(2x)) + 3(t^2 + tx^2 + \frac{1}{12}x^4),$$

which is the exact solution.

Tables 3 and 4 and Figures 3 and 4 show the comparison between exact and approximate solutions of $T(0,t)$ and $T_x(0,t)$ resulted from MVIM, BFM [3], FSM [3, 4], FDM [3, 40] and SFDM [3, 41]. In comparison with the methods in [3, 4, 40, 41] the numerical results show that the MVIM is more accurate.

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Table 3. The comparison between exact and MVIM, BFM [3], FSM [3,4], FDM [3,40] and SFDM [3,41] solutions for $T(0,t)$.

t	Exact	MVIM	MVIM	BFM[3]	FSM[3,4]	FDM[3,40]	SFDM[3,41]
-	-	n=3	n=5	-	-	-	-
0.1	1.3706400	1.3706400	1.3706400	1.3120944	1.4189298	1.3706442	1.3857342
0.3	0.8723884	0.8723884	0.8723884	0.8422776	0.8980560	0.8723989	0.8802165
0.5	1.0206705	1.0206705	1.0206705	1.0021023	1.0432764	1.0206703	1.0243216
0.7	1.5916201	1.5916201	1.5916201	1.5782382	1.6225598	1.5916133	1.5932585
0.9	2.4846474	2.4846474	2.4846474	2.4735959	2.5316474	2.4846480	2.4855032

Table 4. The comparison between exact and MVIM, BFM [3], FSM [3, 4], FDM [3, 40] and SFDM [3, 41] solutions for $T_x(0,t)$

t	Exact	MVIM	MVIM	BFM[3]	FSM[3,4]	FDM[3,40]	SFDM[3,41]
-	-	n=3	n=5	-	-	-	-
0.1	2.6812801	2.6812813	2.6812801	2.6811416	2.6047118	2.6811416	2.6047118
0.3	1.2047768	1.2047773	1.2047768	1.2047477	1.1675152	1.2047477	1.1675152
0.5	0.5413411	0.5413413	0.5413411	0.5413436	0.5240925	0.5413436	0.5240925
0.7	0.2432402	0.2432403	0.2432402	0.2432393	0.2354243	0.2432393	0.2354243
0.9	0.1092948	0.1092949	0.1092948	0.1092749	0.1053256	0.1092749	0.1053256

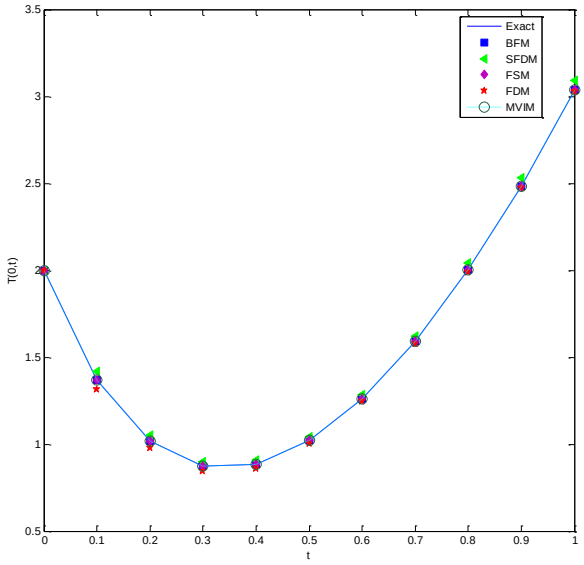

Figure 4. The comparison between exact and MVIM, BFM [3], FSM [3,4], FDM [3,40] and SFDM [3,41] solutions for $T(0,t)$

Example 3. In this example let us consider the following IHCP:

$$T_t(x,t) = T_{xx}(x,t) - x\exp(-t); \qquad 0 < x < 1, \quad 0 < t < 1,$$

$$T(x,0) = x + x^2; \qquad 0 \le x \le 1,$$

$$T(0,t) = p(t); \qquad 0 \le t \le 1,$$

$$T(1,t) = 2t + \exp(-t) + 1; \qquad 0 \le t \le 1,$$

$$T(0.2,t) = 2t + 0.5\exp(-t) + 0.25\cos(0.5); \qquad 0 \le t \le 1.$$

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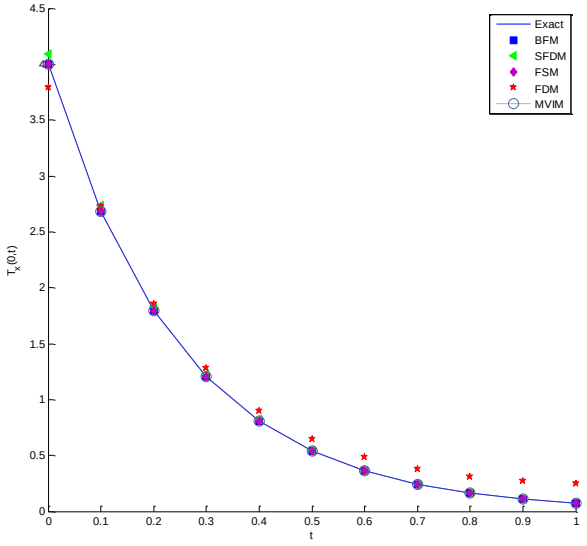

Figure 5. The comparison between exact and MVIM, BFM [3], FSM [3, 4], FDM [3, 40] and SFDM [3, 41] solutions for $T_x(0,t)$.

The exact solution of this problem is:

$$T(x,t) = 2t + x\exp(-t) + x^2, \qquad T_x(x,t) = \exp(-t) + 2x.$$

Consider $T_0 = T(0.5,t) + (x-0.5)T_x(0.5,t)$, where $T_x(0.5,t) = \exp(-t) + 1$. Then we obtain:

$$T_1 = T_0 + \int_{x_1}^{x} (s-x)(T_{0_{ss}} - T_{0_t})ds$$

$$= T_0 + (x-0.5)^2 = 2t + x\exp(-t) + x^2.$$

which is the exact solution.

Conclusion

In this paper, a modified variational iteration method was successfully applied to solve the inverse heat conduction problem. This method solves the problem without any discretization of variables. Thus, it is not affected by rounding errors in the computational process. Application of MVIM is easy and straightforward.

Using the MVIM, a function series is obtained which converges to the exact solution of the discussed problem. In comparison with the methods in [3, 4, 40, 41], the numerical results show that the MVIM is more accurate.

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