A Nonholonomic Mechanical Structure for the Two-Dimensional Monolayer Systems

Esmaeil Azizpour, Ghazaleh Moazzami; University of Guilan
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Abstract

In this paper we study the dynamics of the 2D-motion of a particle of monolayer. A geometric approach to nonholonomic constrained mechanical systems is applied to a problem from the two dimensional geometric dynamics of the Langmuir-Blodgett monolayer. We consider a constraint submanifold of the 1-jet space $J^1(T, R^2)$, corresponding to the given constraint condition in a monolayer space and construct the corresponding constrained mechanical system on the constraint submanifold. Then the equation of motion defined on the constraint submanifold is presented.

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Introduction

Physical systems are often subjected to various types of constraints. Generally, these constraints are position, or geometric (holonomic), constraints, or velocity (nonholonomic) constraints. If a system has constraint equations that involve velocities or derivatives of system coordinates, the constraint equations are said to be nonholonomic and the mechanical system is said to be a nonholonomic system. Although almost all the work on nonholonomic systems is concerned with the case of constraints linear in components of velocities, but a geometric theory covering general nonholonomic systems has been developed by Krupkova [2], [3].

An example of such a system might be monolayer systems. In this paper we present geometric concept of the theory of nonholonomic mechanical systems developed by

*Corresponding author: eazizpour@guilan.ac.ir
Krupkova [3] and consider the application of this theory to a problem from the two-dimensional geometric dynamics of the Langmuir-Blodgett monolayer [4]. To this aim, we consider a constraint submanifold of the 1-jet space $J^1(T, R^2)$ corresponding to the given constraint conditions and construct the corresponding constrained mechanical system on the constraint submanifold. Then the equations of motion defined on the constraint submanifold are presented.

Section 3 of this paper contains a brief outline of basic geometrical concepts of nonholonomic mechanical systems that we will need. For proofs and more details see [2], [3].

**Nonholonomic mechanical systems in a monolayer space**

In this section we begin with a brief introduction to a monolayer space [1], [5] and then describe the nonholonomic mechanical system related to it. To obtain the 2D-motion equation of a particle of monolayer, we define a first order mechanical system [α] in this space and calculate the nonholonomic constrained system [α_q] related to the mechanical system [α].

We start with the usual physical time defined by the Euclidian manifold $(T = [0, \infty))$ and we also consider the plane manifold $R^2$ having the polar coordinates $(r, \varphi)$, where $r > 0$ and $\varphi \in [0, 2\pi)$, and construct the 1-jet vector bundle $J^1(T, R^2) \rightarrow R \times R^2$, locally endowed with the coordinates $(t, q^1, q^2, \dot{q}^1, \dot{q}^2) = (t, r, \varphi, \dot{r}, \dot{\varphi})$.

Using the special function:

$$f(z) = -\int_{-z}^{\infty} \frac{e^{-t}}{t} dt$$

we study the 2D-motion of a particle of monolayer in plane moving in such a way, that the square of its speed decreases proportionally to the reciprocal value of time passed from the beginning of the motion.

We denote by $(t)$ the coordinate on $X = T$, by $(t, r, \varphi)$ fibered coordinates on $Y = T \times R^2$, and by $(t, r, \varphi, \dot{r}, \dot{\varphi})$ the associated coordinates on $J^1(T, R^2)$.

This particle of monolayer governed by the jet Lagrangian function $L: J^1(T, R^2) \rightarrow R$ defined by

$$L(t, r, \dot{r}, \dot{\varphi}) = \frac{m}{2} \dot{r}^2 + \frac{mr^2}{2} \dot{\varphi}^2 - pr^5 |V| e^{\frac{2|V|t}{r}} \cdot \hat{r}^{-1} + U(t, r)$$

(1)
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where we have the following physical meanings: \( m \) is the mass of the particle; \( V \) is the LB-monolayer compressing speed; \( p \) is a constant monolayer parameter given by the physical formula:

\[
p = \frac{\pi^2 q^2 \rho_0^2}{\varepsilon \varepsilon_0 R_0^2};
\]

\( U_s(t, r) \) is an electro capillarity potential energy including the monomolecular layer function:

\[
U(t, r) = p \left\{ -\frac{4}{3} r^5 + \frac{16}{15} (|V| t)^4 + \frac{1}{30} (|V| t)^2 r^3 + \frac{1}{45} (|V| t)^3 r^2 + \frac{1}{45} (|V| t)^4 r \\
+ \frac{2}{45} (|V| t)^5 \right\} e^{\frac{|V| t}{r}} - \frac{4}{45} \left( \frac{|V| t}{r} \right)^6 f \left( \frac{|V| t}{r} \right).
\]

According to the Lagrange theory on fibered manifolds [3], to every Lagrangian there exists a unique Lepagean equivalent, denoted by \( \theta_\lambda \), where \( \lambda \) is a Lagrangian on \( J^1(T, R^2) \). The Euler-Lagrange form defined by the Lepagean equivalent depends only on the Lagrangian \( \lambda \). Therefore it is referred to as the Euler-Lagrange form of the Lagrangian \( \lambda \), and is denoted by \( E_\lambda \). Now, we shall find chart expressions for the Lepagean and the Euler-Lagrange forms.

In fibered coordinates \((t, r, \varphi, r', \varphi')\) on \( J^1(T, R^2) \), the module of contact 1-forms (denoted by \( \Omega^1 J^1(T, R^2) \)) is locally generated by the forms \( \omega^1 = dr - r' dt \) and \( \omega^2 = d\varphi - \varphi' dt \). Put \( \lambda = \lambda dt \) and denote

\[
\theta_\lambda = \lambda + \frac{\partial L}{\partial \dot{r}} \omega^1 + \frac{\partial L}{\partial \dot{\varphi}} \omega^2,
\]

accordingly

\[
\theta_\lambda = \frac{1}{2} \left\{ m \left( \dot{r}^2 + r^2 \dot{\varphi}^2 \right) - p r^5 |V| e^{\frac{|V| t}{r}} \dot{r}^{-1} + U(t, r) \right\} dt \\
+ \left( m \dot{r} + p r^5 |V| e^{\frac{|V| t}{r}} \dot{r}^{-2} \right) \omega^1 + (m r^2 \dot{\varphi}) \omega^2.
\]

We define a first order mechanical system \([\alpha]\) on the fibered manifold \( J^1(T, R^2) \to R \times R^2 \) represented by the 2-form with respect to (15)

\[
\alpha = d\theta_\lambda + F = \left( m r \dot{\varphi}^2 - \frac{5 p r^4 |V| e^{\frac{|V| t}{r}}}{\dot{r}} + \frac{2 p r^3 |V|^2 t e^{\frac{|V| t}{r}}}{\dot{r}^2} + \frac{\partial u}{\partial t} \right) dr \wedge dt \\
+ \left( \frac{5 p r^4 |V| e^{\frac{|V| t}{r}}}{\dot{r}^2} - \frac{2 p r^3 |V|^2 t e^{\frac{|V| t}{r}}}{\dot{r}^2} \right) dt \wedge \omega^1 + \left( \frac{5 p r^4 |V| e^{\frac{|V| t}{r}}}{\dot{r}^2} - \frac{2 p r^3 |V|^2 t e^{\frac{|V| t}{r}}}{\dot{r}^2} \right) dr \wedge \omega^1 \\
+ \left( m - \frac{2 p r^5 |V| e^{\frac{|V| t}{r}}}{r^3} \right) dt \wedge \omega^1 + (2 m r \dot{\varphi}) dr \wedge \omega^2 + (m r^2) d\varphi \wedge \omega^2 + F.
\]
This mechanical system is related to the dynamical form with respect to (11)

\[ E = E_1 \, dr \wedge dt + E_2 \, d\varphi \wedge dt, \]  

where

\[
E_1 = \left( m \ddot{r} \phi - \frac{10pr^4 |V|e^{\frac{2|V|l}{r}}}{\dot{r}} + \frac{4pr^3 |V|^2 e^{\frac{2|V|l}{r}}}{\dot{r}} - \frac{2pr^4 |V|^2 e^{\frac{2|V|l}{r}}}{\dot{r}^2} + \frac{\partial U}{\partial r} \right) 
- \left( m + \frac{2pr^5 |V|e^{\frac{2|V|l}{r}}}{\dot{r}^2} \right) \dot{r} \, \ddot{r},
\]

and

\[ E_2 = -(2mrr' + mr^2 \varphi). \]

We consider the nonholonomic constraint \( Q \) given by the equation

\[ f (t, r, \varphi, r'; \varphi) \equiv [(r)^2 + (\varphi)^2] - \frac{1}{t} = 0, \]  

which means that the particle's speed decreases proportionally to \( \frac{1}{\sqrt{t}} \). In a neighborhood of the submanifold \( Q \)

\[ \text{rank} \left( \frac{\partial f}{\partial r}, \frac{\partial f}{\partial \varphi} \right) = \text{rank} \left( 2\dot{r}, 2\dot{\varphi} \right) = 1. \]  

Let \( U \subset J^1 (T, R^2) \) be the set of all points, where \( \phi > 0 \), and consider on \( U \) the adapted coordinates \((t, r, \varphi, r', \tilde{t})\), where \( \tilde{t} = \dot{\varphi} - g \), \( g = \sqrt{\frac{1}{t} - (\dot{r})^2} \) is the equation of the constraint (6) in normal form.

The constrained system \([\alpha_Q] \) related to the mechanical system \([\alpha] \) and the constraint \( Q \) is the equivalence class of the 2-form (with respect to (18))

\[ \alpha_Q = A'_1 \omega^1 \wedge dt + B'_{1,1} \omega^1 \wedge d\dot{r} + \Phi + \Phi_{(2)} \]  

on \( Q \), where \( \Phi \) is any 2-contact 2-form and \( \Phi_{(2)} \) is any constraint 2-form defined on \( Q \).

Calculating \( \mathcal{L} \) and calculating \( A'_1, B'_{1,1} \) by relationships (19), (20)

\[ \mathcal{L}(t, r, \varphi, \dot{r}, \dot{\varphi}) = L \left( t, r, \varphi, \dot{r}, \sqrt{\frac{1}{t} - (\dot{r})^2} \right). \]

Then

\[ \mathcal{L} = \frac{m}{2} \dot{r}^2 + \frac{mr^2}{2} \left( \frac{1}{t} - \dot{r}^2 \right) - \frac{pr^5 |V|e^{\frac{2|V|l}{r}}}{\dot{r}} \cdot \dot{r}^{-1} + U(t, r), \]

and

\[ A'_1 = mr^2 + 2mr \dot{r} - \frac{m \ddot{r}^2 g^2}{2t^2} - \frac{10pr^4 |V|e^{\frac{2|V|l}{r}}}{\dot{r}} + \frac{4pr^3 |V|^2 e^{\frac{2|V|l}{r}}}{\dot{r}} - \frac{2pr^4 |V|^2 e^{\frac{2|V|l}{r}}}{\dot{r}^2} + \frac{\partial U}{\partial r}. \]
Reduced equation of motion of the constrained system is as follows

\[ [A'_i + B'_{1,i} r] \cdot J^2 \breve{\gamma} = 0, \]  

where \( \breve{\gamma} = (t, r(t), \varphi(t)) \) is a section satisfying the constraint equation \( f \circ J^1 \gamma = 0 \).

### Lagrangian systems on fibered manifolds

Throughout this section we consider a fibered manifold \( \pi: X \rightarrow Y \) with a one dimensional base space \( X \) and \( (m+1) \)-dimensional total space \( Y \). We use jet prolongations \( \pi_1 : J^1(X,Y) \rightarrow X \) and \( \pi_2 : J^2(X,Y) \rightarrow X \) and jet projections \( \pi_{1,0} : J^1(X,Y) \rightarrow Y \) and \( \pi_{2,1} : J^2(X,Y) \rightarrow J^1(X,Y) \). Configuration space at a fixed time is represented by a fiber of the fibered manifold \( \pi \) and a corresponding phase space is then a fiber of the fibered manifold \( \pi_1 \). Local fibered coordinates on \( Y \) are denoted by \( (t, q^\sigma) \), where \( 1 \leq \sigma \leq m \). The associated coordinates on \( J^1(X,Y) \) and \( J^2(X,Y) \) are denoted by \( (t, q^\sigma, \dot{q}^\sigma) \) and \( (t, q^\sigma, \dot{q}^\sigma, \ddot{q}^\sigma) \), respectively. In calculations we use either a canonical basis of one forms on \( J^1(X,Y) \), \( (dt, dq^\sigma, d\dot{q}^\sigma) \), or a basis adapted to the contact structure

\[ (dt, \omega^\sigma, d\dot{q}^\sigma), \]

where

\[ \omega^\sigma = dq^\sigma - \dot{q}^\sigma dt, \quad 1 \leq \sigma \leq m. \]

Whenever possible, the summation convention is used. If \( f(t, q^\sigma, \dot{q}^\sigma) \) is a function defined on an open set of \( J^1(X,Y) \) we write

\[ \frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial q^\sigma} \dot{q}^\sigma + \frac{\partial f}{\partial \dot{q}^\sigma} \ddot{q}^\sigma, \quad \frac{d\dot{f}}{dt} = \frac{\partial \dot{f}}{\partial t} + \frac{\partial \dot{f}}{\partial q^\sigma} \dot{q}^\sigma. \]

A differential form \( \rho \) is called contact if \( J^1 \gamma \cdot \rho = 0 \) for every section \( \gamma \) of \( \pi \). A contact 2-form \( \rho \) is called 1-contact if for every vertical vector field \( \xi \), \( i_\xi \rho_1 \) is a horizontal; \( \rho \) is 2-contact if \( i_\xi \rho_1 \) is 1-contact. The operator assigning to \( \rho \) it’s 1-contact part is denoted by \( p_1 \).

If \( \lambda \) is a Lagrangian on \( J^1(X,Y) \), we denote by \( \theta_\lambda \) its Lepage equivalent or Cartan form and \( E_\lambda \) its Euler-Lagrange form, respectively. Recall that \( E_\lambda = p_1 \, d\theta_\lambda \). In fibered coordinates where \( \lambda = L(t, q^\sigma, \dot{q}^\sigma) \, dt \), we have

\[ \theta_\lambda = L \, dt + \frac{\partial L}{\partial q^\sigma} \omega^\sigma, \quad \text{and} \]

\[ B'_{1,1} = m(r^2 - 1) - \frac{mr^2}{tg^2} + \frac{2pr^2}{r^3} + \frac{2\dot{r}^2}{r^3}. \]
where the components $E_\sigma (L) = \frac{\partial L}{\partial \dot{q}^\sigma} - \frac{d}{dt} \frac{\partial L}{\partial \ddot{q}^\sigma}$, are the Euler-Lagrange expressions.

Since the functions $E_\sigma$ are affine in the second derivatives we write

$$E_\sigma = A_\sigma + B_{\sigma \nu} \dot{q}^\nu,$$

where

$$A_\sigma = \frac{\partial L}{\partial \dot{q}^\sigma} - \frac{\partial^2 L}{\partial t \partial \dot{q}^\sigma} - \frac{\partial^2 L}{\partial \dot{q}^\nu \partial \dot{q}^\sigma} \dddot{q}^\nu, \quad B_{\sigma \nu} = - \frac{\partial^2 L}{\partial \dot{q}^\sigma \partial \ddot{q}^\nu}. \quad (12)$$

A section $\gamma$ of $\pi$ is called a path of the Euler-Lagrange form $E_\lambda$ if

$$E_\lambda \circ J^2 \gamma = 0. \quad (13)$$

In fibered coordinates this equation represents a system of $m$ second-order ordinary differential equations

$$A_\sigma \left( t, \gamma^\nu, \frac{d\gamma^\nu}{dt} \right) + B_{\sigma \rho} \left( t, \gamma^\nu, \frac{d\gamma^\nu}{dt} \right) \frac{d^2 \gamma^\rho}{dt^2} = 0, \quad (14)$$
for components $\gamma^\nu(t)$ of a section $\gamma$, where $1 \leq \nu \leq m$. These equations are called Euler-Lagrange equations or motion equations and their solutions are called paths.

Euler-Lagrange equations (14) can be written in the form

$$J^1 \gamma^\nu i_\xi \alpha = 0,$$

where $\alpha = d\theta_\lambda + F$ is any 2-form defined on an open subset $W \subset J^1(X,Y)$, such that $p_\lambda \alpha = E_\lambda$, and $F$ is a 2-contact 2-form. In fibered coordinates we have $F = F_{\sigma \nu} \omega^\sigma \wedge \omega^\nu$, where $F_{\sigma \nu}(t,q^\rho,\dot{q}^\rho)$ are arbitrary functions. Recall from [3] that the family of all such (local) 2-forms:

$$\alpha = d\theta_\lambda + F = A_\sigma \omega^\sigma \wedge dt + B_{\sigma \nu} \omega^\sigma \wedge dq^\nu + F \quad (15)$$

is called a first order Lagrangian system, and is denoted by $[\alpha]$.

A non-holonomic constrained mechanical system is defined on the $(2m+1-k)$-dimensional constraint submanifold $Q \subset J^1(X,Y)$ fibered over $Y$ and given by $k$ equations

$$f^i(t,q^1,\ldots,q^m,\dot{q}^1,\ldots,\dot{q}^m) = 0, \quad 1 \leq i \leq k,$$

where

$$\text{rank} \left( \frac{\partial f^i}{\partial \dot{q}^\sigma} \right) = k,$$

or following [6], equivalently in an explicit form

$$\dddot{q}^{m-k+i} = g^i(t,q^\sigma,\dot{q}^1,\ddot{q}^2,\ldots,\dot{q}^{m-k}), \quad 1 \leq i \leq k. \quad (17)$$
By a nonholonomic constrained system arising from the Lagrangian system \([\alpha]\) and constraint forms on the constraint submanifold \(Q\), we mean the equivalence class \([\alpha_Q]\) on \(Q\), where

\[
\alpha_Q = \sum_{i=1}^{m-k} A_i' \omega^i \wedge dt + \sum_{s=1}^{m-k} B_{ls}' \omega^l \wedge dq^s + \bar{F} + \phi(2),
\]

where \(F\) is any 2-contact \(\pi_{1,0}\) horizontal 2-form and \(\phi_2\) is any constraint 2-form defined on \(Q\), and \(t\) is the canonical embedding of \(Q\) into \(J^1(X,Y)\). The local form of \([\alpha_Q]\) is

\[
\alpha_Q = \sum_{i=1}^{m-k} A_i' \omega^i \wedge dt + \sum_{s=1}^{m-k} B_{ls}' \omega^l \wedge dq^s + \bar{F} + \phi(2),
\]

where the components \(A_i'\) and \(B_{ls}'\) are given by

\[
A_i' = \frac{\partial \bar{L}}{\partial q^i} + \frac{\partial \bar{L}}{\partial q^{m-k+i}} \frac{\partial g^i}{\partial q^l} - \frac{\partial c}{\partial t} \frac{\partial L}{\partial q^l} + \frac{\partial (\partial L}{\partial q^{m-k+i}} \frac{\partial g^i}{\partial q^l})_l dt \left( \frac{\partial g^i}{\partial q^l} - \frac{\partial g^j}{\partial q^l} \right) - \frac{\partial g^i}{\partial q^{m-k+i}} \frac{\partial g^j}{\partial q^l},
\]

\[
B_{ls}' = - \frac{\partial^2 \bar{L}}{\partial q^l \partial q^s} + \left( \frac{\partial L}{\partial q^{m-k+i}} \right)_l \frac{\partial^2 g^i}{\partial q^l \partial q^s},
\]

\[
\bar{L} = L \circ t, \quad \text{and}
\]

\[
\frac{\partial c}{\partial t} = \frac{\partial q^s}{\partial t} \frac{\partial q^s}{\partial q^s} + g^i \frac{\partial q^{m-k+i}}{\partial q^s}.
\]

The equations of the motion of the constrained system \([\alpha_Q]\) in fibered coordinates take the form

\[
\left( A_i' + \sum_{s=1}^{m-k} B_{ls}' \dot{q}^s \right) \circ J^2 \dot{y} = 0.
\]

for components \(y^1(t), y^2(t), \ldots, y^{m-k}(t)\) of a \(Q\) – admissible section \(\dot{y}\) dependent on time \(t\) and parameters \(q^{m-k+1}, q^{m-k+2}, \ldots, q^m\), which have to be determined as functions \(y^{m-k+1}(t), y^{m-k+2}(t), \ldots, y^m(t)\) from the equations (17) of the constraint

\[
\frac{dq^{m-k+i}}{dt} = g^i \left( t, q^\sigma, \frac{dq^1}{dt}, \frac{dq^2}{dt}, \ldots, \frac{dq^{m-k}}{dt} \right), \quad 1 \leq i \leq k.
\]
References


