A Nonholonomic Mechanical Structure for the Two-
Dimensional Monolayer Systems

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Abstract

In this paper we study the dynamics of the 2D-motion of a particle of monolayer. A
geometric approach to nonholonomic constrained mechanical systems is applied to a
problem from the two dimensional geometric dynamics of the Langmuir-Blodgett
monolayer. We consider a constraint submanifold of the 1-jet space $J^1(T, R^2)$,
corresponding to the given constraint condition in a monolayer space and construct the
corresponding constrained mechanical system on the constraint submanifold. Then the
equation of motion defined on the constraint submanifold is presented.

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Introduction

Physical systems are often subjected to various types of constraints. Generally, these
constraints are position, or geometric (holonomic), constraints, or velocity (nonholono-
mic) constraints. If a system has constraint equations that involve velocities or
derivatives of system coordinates, the constraint equations are said to be nonholonomic
and the mechanical system is said to be a nonholonomic system. Although almost all the
work on nonholonomic systems is concerned with the case of constraints linear in
components of velocities, but a geometric theory covering general nonholonomic
systems has been developed by Krupkova [2], [3].

An example of such a system might be monolayer systems. In this paper we present
g geometric concept of the theory of nonholonomic mechanical systems developed by

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Krupkova [3] and consider the application of this theory to a problem from the two dimensional geometric dynamics of the Langmuir-Blodgett monolayer [4]. To this aim, we consider a constraint submanifold of the 1-jet space $J^1(T,R^2)$ corresponding to the given constraint conditions and construct the corresponding constrained mechanical system on the constraint submanifold. Then the equations of motion defined on the constraint submanifold are presented.

Section 3 of this paper contains a brief outline of basic geometrical concepts of nonholonomic mechanical systems that we will need. For proofs and more details see [2], [3].

**Nonholonomic mechanical systems in a monolayer space**

In this section we begin with a brief introduction to a monolayer space [1], [5] and then describe the nonholonomic mechanical system related to it. To obtain the 2D-motion equation of a particle of monolayer, we define a first order mechanical system $\alpha$ in this space and calculate the nonholonomic constrained system $[\alpha_0]$ related to the mechanical system $[\alpha]$.

We start with the usual physical time defined by the Euclidean manifold $(T = [0, \infty))$ and we also consider the plane manifold $R^2$ having the polar coordinates $(r, \varphi)$, where $r > 0$ and $\varphi \in [0, 2\pi)$, and construct the 1-jet vector bundle $J^1(T, R^2) \to R \times R^2$, locally endowed with the coordinates $(t, q^1, q^2, \dot{q}^1, \dot{q}^2) = (t, r, \varphi, \dot{r}, \dot{\varphi})$.

Using the special function:

$$f(z) = -\int_{-z}^{\infty} \frac{e^{-t}}{t} dt$$

we study the 2D-motion of a particle of monolayer in plane moving in such a way, that the square of its speed decreases proportionally to the reciprocal value of time passed from the beginning of the motion.

We denote by $(t)$ the coordinate on $X = T$, by $(t, r, \varphi)$ fibered coordinates on $Y = T \times R^2$, and by $(t, r, \varphi, \dot{r}, \dot{\varphi})$ the associated coordinates on $J^1(T, R^2)$.

This particle of monolayer governed by the jet Lagrangian function $L: J^1(T, R^2) \to R$ defined by

$$L(t, r, \dot{r}, \dot{\varphi}) = \frac{m}{2} \dot{r}^2 + \frac{m r^2}{2} \dot{\varphi}^2 - p r^5 |V| e^{\frac{2i\varphi}{r}} \cdot \dot{r}^{-1} + U(t, r),$$

(1)
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where we have the following physical meanings: \( m \) is the mass of the particle; \( V \) is the \( LB\)-monolayer compressing speed; \( p \) is a constant monolayer parameter given by the physical formula:

\[
p = \frac{\pi^2 q^2}{\varepsilon \varepsilon_0} \frac{\rho_0^2}{R_0^2};
\]

\( U_s(t, r) \) is an electro capillarity potential energy including the monomolecular layer function:

\[
U(t, r) = p \left\{ -\frac{4}{3} r^{-5} + \frac{16}{15} (|V| t) r^4 + \frac{1}{30} (|V| t)^2 r^3 + \frac{1}{45} (|V| t)^3 r^2 + \frac{1}{45} (|V| t)^4 r \\
+ \frac{2}{45} (|V| t)^5 \right\} e^{\frac{2|V| t}{r}} - \frac{4}{45} \frac{(|V| t)^6}{r} f \left( \frac{2|V| t}{r} \right).
\]

According to the Lagrange theory on fibered manifolds [3], to every Lagrangian there exists a unique Lepagean equivalent, denoted by \( \theta_\lambda \), where \( \lambda \) is a Lagrangian on \( J^1(T, R^2) \). The Euler-Lagrange form defined by the Lepagean equivalent depends only on the Lagrangian \( \lambda \). Therefore it is referred to as the Euler-Lagrange form of the Lagrangian \( \lambda \), and is denoted by \( E_\lambda \). Now, we shall find chart expressions for the Lepagean and the Euler-Lagrange forms.

In fibered coordinates \( (t, r, \varphi, \varphi', \varphi') \) on \( J^1(T, R^2) \), the module of contact 1-forms (denoted by \( \Omega^1 J^1(T, R^2) \)) is locally generated by the forms \( \omega^1 = dt - r'dt \) and \( \omega^2 = d\varphi - \varphi'dt \). Put \( \lambda = L dt \) and denote

\[
\theta_\lambda = L dt + \frac{\partial L}{\partial r} \omega^1 + \frac{\partial L}{\partial \varphi} \omega^2,
\]

accordingly

\[
\theta_\lambda = \left( \frac{1}{2} m \left( \dot{r}^2 + r^2 \dot{\varphi}^2 \right) - p r^5 |V| e^{\frac{2|V| t}{r}} \cdot \dot{r}^{-1} + U(t, r) \right) dt \\
+ \left( m \dot{r} + p r^5 |V| e^{\frac{2|V| t}{r}} \cdot \dot{r}^{-2} \right) \omega^1 + (m r^2 \dot{\varphi}) \omega^2.
\]

We define a first order mechanical system \( [\alpha] \) on the fibered manifold \( J^1(T, R^2) \to R \times R^2 \) represented by the 2-form with respect to (15)

\[
\alpha = d\theta_\lambda + F = \left( m r \dot{\varphi}^2 - \frac{5p r^4 |V| e^{\frac{2|V| t}{r}}}{r} + \frac{2pr^3 |V|^2 t e^{\frac{2|V| t}{r}}}{r} + \frac{\partial U}{\partial r} \right) dr \wedge dt \\
+ \left( \frac{2pr^4 |V|^2 e^{\frac{2|V| t}{r}}}{r^2} \right) dt \wedge \omega^1 + \left( \frac{5pr^4 |V|^2 e^{\frac{2|V| t}{r}}}{r^2} - \frac{2pr^3 |V|^2 t e^{\frac{2|V| t}{r}}}{r^2} \right) dr \wedge \omega^1 \\
+ \left( m - \frac{2pr^5 |V| e^{\frac{2|V| t}{r}}}{r^3} \right) d\varphi \wedge \omega^1 + (2mr \dot{\varphi}) dr \wedge \omega^2 + (mr^2 d\varphi \wedge \omega^2 + F. \quad (4)
\]
This mechanical system is related to the dynamical form with respect to (11)

\[ E = E_1 \, dr \wedge dt + E_2 \, d\varphi \wedge dt, \]  

where

\[
E_1 = \left( mr\dot{\varphi} - \frac{10pr^4 |V|e^{\frac{2|V|t}{r}}}{r} + 4pr^3 \frac{e^{\frac{2|V|t}{r}}}{\dot{r}} - \frac{2pr^4 |V|^2 e^{\frac{2|V|t}{r}}}{\dot{r}^2} + \frac{\partial U}{\partial r} \right)
- \left( m + \frac{2pr^5 |V|e^{\frac{2|V|t}{r}}}{\dot{r}^2} \right) r^2,
\]

and

\[ E_2 = -(2mr\dot{r} + mr^2 \varphi). \]

We consider the nonholonomic constraint \( Q \) given by the equation

\[ f (t,r,\varphi,\dot{r},\ddot{\varphi}) \equiv \left[ (r)^2 + (\varphi)^2 \right] - \frac{1}{t} = 0, \tag{6} \]

which means that the particle's speed decreases proportionally to \( \frac{1}{\sqrt{t}} \). In a neighborhood of the submanifold \( Q \)

\[
\text{rank} \left( \frac{\partial f}{\partial r} , \frac{\partial f}{\partial \varphi} \right) = \text{rank} \left( 2\dot{r} , 2\dot{\varphi} \right) = 1. \tag{7}
\]

Let \( U \subset J^1 (T, R^2) \) be the set of all points, where \( \varphi > 0 \), and consider on \( U \) the adapted coordinates \((t,r,\varphi,\dot{r},\ddot{\varphi})\), where \( \dddot{\varphi} = \dot{\varphi} - g, \ g = \frac{1}{\sqrt{t}} - (\dot{r})^2 \) is the equation of the constraint (6) in normal form.

The constrained system \([\alpha_Q]\) related to the mechanical system \([\alpha]\) and the constraint \( Q \) is the equivalence class of the 2-form (with respect to (18))

\[ \alpha_Q = A'_1 \omega^1 \wedge dt + B'_{1,1} \omega^1 \wedge d\dot{r} + \bar{F} + \phi_{(2)} \]  

on \( Q \), where \( \bar{F} \) is any 2-contact 2-form and \( \phi_{(2)} \) is any constraint 2-form defined on \( Q \).

Calculating \( \bar{L} = L \circ \tau \) and calculating \( A'_1, B'_{1,1} \) by relationships (19), (20)

\[ \bar{L}(t,r,\varphi,\dot{r},\ddot{\varphi}) = L \left( t,r,\varphi, \dot{r}, \sqrt{\frac{1}{t} - (\dot{r})^2} \right). \]

Then

\[ \bar{L} = \frac{m}{2} \dddot{\varphi}^2 + \frac{mr^2}{2} \left( \frac{1}{t} - \dot{r}^2 \right) - pr^5 |V|e^{\frac{2|V|t}{r}} \cdot \dddot{r}^{-1} + U(t,r), \]

and

\[ A'_1 = mr g^2 + 2mr^2 \dddot{\varphi} - \frac{m \dddot{r}^2}{2t^2} g^2 - \frac{10pr^4 |V|e^{\frac{2|V|t}{r}}}{r} + 4pr^3 \frac{e^{\frac{2|V|t}{r}}}{\dot{r}} - \frac{2pr^4 |V|^2 e^{\frac{2|V|t}{r}}}{\dot{r}^2} + \frac{\partial U}{\partial r}, \]
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\[ B_{1,1} = m(r^2 - 1) - \frac{m r^2}{t g^2} + 2 p r^2 t |v| e^{\frac{2 |v| t}{r^3}}. \]

Reduced equation of motion of the constrained system is as follows

\[ [A^1 + B_{1,1}' r] \circ J^2 \bar{y} = 0, \tag{9} \]

where \( \bar{y} = (t, r(t), \varphi(t)) \) is a section satisfying the constraint equation \( f \circ J^1 \gamma = 0 \).

**Lagrangian systems on fibered manifolds**

Throughout this section we consider a fibered manifold \( \pi: Y \to X \) with a one dimensional base space \( X \) and \( (m + 1) \)–dimensional total space \( Y \). We use jet prolongations \( \pi_1: J^1(X,Y) \to X \) and \( \pi_2: J^2(X,Y) \to X \) and jet projections \( \pi_{1,0}: J^1(X,Y) \to Y \) and \( \pi_{2,1}: J^2(X,Y) \to J^1(X,Y) \). Configuration space at a fixed time is represented by a fiber of the fibered manifold \( \pi \) and a corresponding phase space is then a fiber of the fibered manifold \( \pi_1 \). Local fibered coordinates on \( Y \) are denoted by \((t, q^\sigma)\), where \( 1 \leq \sigma \leq m \). The associated coordinates on \( J^1(X,Y) \) and \( J^2(X,Y) \) are denoted by \((t, q^\sigma, \dot{q}^\sigma)\) and \((t, q^\sigma, \dot{q}^\sigma, q^\eta)\), respectively. In calculations we use either a canonical basis of one forms on \( J^1(X,Y) \), \((dt, dq^\sigma, d\dot{q}^\sigma)\), or a basis adapted to the contact structure

\( (dt, \omega^\sigma, d\dot{q}^\sigma) \),

where

\[ \omega^\sigma = dq^\sigma - \dot{q}^\sigma dt, \quad 1 \leq \sigma \leq m. \]

Whenever possible, the summation convention is used. If \( f(t, q^\sigma, \dot{q}^\sigma) \) is a function defined on an open set of \( J^1(X,Y) \) we write

\[ \frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial q^\sigma} \dot{q}^\sigma + \frac{\partial f}{\partial \dot{q}^\sigma} \ddot{q}^\sigma, \quad \frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial q^\sigma} \dot{q}^\sigma. \]

A differential form \( \rho \) is called **contact** if \( J^1 \gamma^* \rho = 0 \) for every section \( \gamma \) of \( \pi \). A contact 2 -form \( \rho \) is called **1-contact** if for every vertical vector field \( \xi \), \( i_{\xi} \rho_1 \) is a horizontal; \( \rho \) is 2 -contact if \( i_{\xi} \rho_1 \) is 1-contact. The operator assigning to \( \rho \) it’s 1-contact part is denoted by \( p_1 \).

If \( \lambda \) is a Lagrangian on \( J^1(X,Y) \), we denote by \( \Theta_\lambda \) its **Lepage equivalent** or **Cartan form** and \( E_\lambda \) its Euler-Lagrange form, respectively. Recall that \( E_\lambda = p_1 d\Theta_\lambda \). In fibered coordinates where \( \lambda = L(t, q^\sigma, \dot{q}^\sigma) dt \), we have

\[ \Theta_\lambda = L dt + \frac{\partial L}{\partial q^\sigma} \omega^\sigma, \tag{10} \]

and
where the components \( E_{\sigma} (L) = \frac{\partial L}{\partial \dot{q}^\sigma} - \frac{d}{dt} \frac{\partial L}{\partial \ddot{q}^\sigma} \), are the Euler-Lagrange expressions. Since the functions \( E_{\sigma} \) are affine in the second derivatives we write

\[
E_{\sigma} = A_{\sigma} + B_{\sigma \nu} \dot{q}^\nu,
\]

where

\[
A_{\sigma} = \frac{\partial L}{\partial \dot{q}^\sigma} - \frac{\partial^2 L}{\partial t \partial \dot{q}^\sigma} - \frac{\partial^2 L}{\partial \dot{q}^\nu \partial \dot{q}^\sigma} \ddot{q}^\nu, \quad B_{\sigma \nu} = -\frac{\partial^2 L}{\partial \dot{q}^\sigma \partial \dot{q}^\nu}.
\]

A section \( \gamma \) of \( \pi \) is called a path of the Euler-Lagrange form \( E_\lambda \) if

\[
E_\lambda \circ J^2 \gamma = 0.
\]

In fibered coordinates this equation represents a system of \( m \) second-order ordinary differential equations

\[
A_{\sigma} \left( t, \gamma^\nu, \frac{d\gamma^\nu}{dt} \right) + B_{\sigma \nu} \left( t, \gamma^\nu, \frac{d\gamma^\nu}{dt} \right) \frac{d^2 \gamma^\nu}{dt^2} = 0,
\]

for components \( \gamma^\nu(t) \) of a section \( \gamma \), where \( 1 \leq \nu \leq m \). These equations are called Euler-Lagrange equations or motion equations and their solutions are called paths.

Euler-Lagrange equations (14) can be written in the form

\[
J^1 \gamma^\nu \xi^\nu \alpha = 0,
\]

where \( \alpha = d\theta_\lambda + F \) is any 2-form defined on an open subset \( W \subset J^1(X,Y) \), such that \( p_\alpha = E_\lambda \), and \( F \) is a 2-contact 2-form. In fibered coordinates we have \( F = F_{\sigma \nu} \omega^\sigma \wedge \omega^\nu \), where \( F_{\sigma \nu}(t, q^\rho, \dot{q}^\rho) \) are arbitrary functions. Recall from [3] that the family of all such (local) 2-forms:

\[
\alpha = d\theta_\lambda + F = A_{\sigma} \omega^\sigma \wedge dt + B_{\sigma \nu} \omega^\sigma \wedge dq^\nu + F
\]

is called a first order Lagrangian system, and is denoted by \([\alpha]\).

A non-holonomic constrained mechanical system is defined on the \((2m+1-k)\)-dimensional constraint submanifold \( Q \subset J^1(X,Y) \) fibered over \( Y \) and given by \( k \) equations

\[
f^i(t, q^1, \ldots, q^m, \dot{q}^1, \ldots, \dot{q}^m) = 0, \quad 1 \leq i \leq k,
\]

where

\[
\text{rank} \left( \frac{\partial f^i}{\partial \dot{q}^\sigma} \right) = k,
\]

or following [6], equivalently in an explicit form

\[
\dot{q}^{m-k+i} = g^i(t, q^\sigma, \dot{q}^1, \dot{q}^2, \ldots, \dot{q}^{m-k}), \quad 1 \leq i \leq k.
\]
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By a nonholonomic constrained system arising from the Lagrangian system \([\alpha]\) and constraint forms on the constraint submanifold \(Q\), we mean the equivalence class \([\alpha_Q]\) on \(Q\), where

\[\alpha_Q = i^* d\theta_\lambda + \vec{F} + \phi_2(2),\]

where \(\vec{F}\) is any 2-contact \(\pi_{1,0}\) horizontal 2-form and \(\phi_2\) is any constraint 2-form defined on \(Q\), and \(i\) is the canonical embedding of \(Q\) into \(J^1(X,Y)\). The local form of \([\alpha_Q]\) is

\[\alpha_Q = \sum_{l=1}^{m-k} A_l^i \omega^l \wedge dt + \sum_{l,s=1}^{m-k} B_{l,s}^i \omega^l \wedge dq^s + \vec{F} + \phi_2(2), \tag{18}\]

where the components \(A_l^i\) and \(B_{l,s}^i\) are given by

\[A_l^i = \frac{\partial \bar{L}}{\partial q^l} + \frac{\partial \bar{L}}{\partial q^{m-k+i}} \frac{\partial g^i}{\partial q^l} - \frac{\partial c}{\partial t} \frac{\partial \bar{L}}{\partial q^l} + \left(\frac{\partial \bar{L}}{\partial q^{m-k+i}}\right)_t \left[ \frac{\partial g^i}{\partial q^l} - \frac{\partial g^j}{\partial q^l} \frac{\partial g^i}{\partial q^j} \right],\]

\[B_{l,s}^i = -\frac{\partial^2 \bar{L}}{\partial q^l \partial q^s} + \left(\frac{\partial \bar{L}}{\partial q^{m-k+i}}\right)_t \frac{\partial^2 g^i}{\partial q^l \partial q^s} \]

\[\bar{L} = L \circ i, \quad \text{and} \quad \frac{\partial c}{\partial t} = \frac{\partial q^s}{\partial \bar{q}^s} + \frac{\partial g^i}{\partial q^{m-k+i}} \frac{\partial}{\partial q^l} \frac{\partial g^i}{\partial q^{m-k+i}}.\]

The equations of the motion of the constrained system \([\alpha_Q]\) in fibered coordinates take the form

\[\left( A_l^i + \sum_{s=1}^{m-k} B_{l,s}^i \dot{q}^s \right) \circ J^2 \bar{y} = 0. \tag{19}\]

for components \(y^1(t), y^2(t), \ldots, y^{m-k}(t)\) of a \(Q\) - admissible section \(\bar{y}\) dependent on time \(t\) and parameters \(q^{m-k+1}, q^{m-k+2}, \ldots, q^m\), which have to be determined as functions \(y^{m-k+1}(t), y^{m-k+2}(t), \ldots, y^m(t)\) from the equations (17) of the constraint

\[\frac{dq^{m-k+i}}{dt} = g^i \left( t, q^\sigma, \frac{dq^1}{dt}, \frac{dq^2}{dt}, \ldots, \frac{dq^{m-k}}{dt} \right), \quad 1 \leq i \leq k.\]
References


