A Nonholonomic Mechanical Structure for the Two-Dimensional Monolayer Systems

Esmaeil Azizpour, Ghazaleh Moazzami;
University of Guilan

Received: 26 Dec 2014 Revised: 17 Sep 2017

Abstract

In this paper we study the dynamics of the 2D-motion of a particle of monolayer. A geometric approach to nonholonomic constrained mechanical systems is applied to a problem from the two dimensional geometric dynamics of the Langmuir-Blodgett monolayer. We consider a constraint submanifold of the 1-jet space \( J^1(T, R^2) \), corresponding to the given constraint condition in a monolayer space and construct the corresponding constrained mechanical system on the constraint submanifold. Then the equation of motion defined on the constraint submanifold is presented.

Keywords. Nonholonomic constraints, 2D-monolayer Lagrangian.

Mathematical Classification subject: 53C80,70G45.

Introduction

Physical systems are often subjected to various types of constraints. Generally, these constraints are position, or geometric (holonomic), constraints, or velocity (nonholonomic) constraints. If a system has constraint equations that involve velocities or derivatives of system coordinates, the constraint equations are said to be nonholonomic and the mechanical system is said to be a nonholonomic system. Although almost all the work on nonholonomic systems is concerned with the case of constraints linear in components of velocities, but a geometric theory covering general nonholonomic systems has been developed by Krupkova [2], [3].

An example of such a system might be monolayer systems. In this paper we present geometric concept of the theory of nonholonomic mechanical systems developed by

*Corresponding author: eazizpour@guilan.ac.ir
Krupkova [3] and consider the application of this theory to a problem from the two dimensional geometric dynamics of the Langmuir-Blodgett monolayer [4]. To this aim, we consider a constraint submanifold of the 1-jet space \( J^1(T, R^2) \) corresponding to the given constraint conditions and construct the corresponding constrained mechanical system on the constraint submanifold. Then the equations of motion defined on the constraint submanifold are presented.

Section 3 of this paper contains a brief outline of basic geometrical concepts of nonholonomic mechanical systems that we will need. For proofs and more details see [2], [3].

**Nonholonomic mechanical systems in a monolayer space**

In this section we begin with a brief introduction to a monolayer space [1], [5] and then describe the nonholonomic mechanical system related to it. To obtain the 2D-motion equation of a particle of monolayer, we define a first order mechanical system \([\alpha]\) in this space and calculate the nonholonomic constrained system \([\alpha_\varphi]\) related to the mechanical system \([\alpha]\).

We start with the usual physical time defined by the Euclidian manifold \((T = [0, \infty])\) and we also consider the plane manifold \(R^2\) having the polar coordinates \((r, \varphi)\), where \(r > 0\) and \(\varphi \in [0, 2\pi]\), and construct the 1-jet vector bundle \(J^1(T, R^2) \to R \times R^2\), locally endowed with the coordinates \((t, q^1, q^2, \dot{q}^1, \dot{q}^2) = (t, r, \varphi, \dot{r}, \dot{\varphi})\).

Using the special function:

\[
f(z) = -\int_{-z}^{\infty} \frac{e^{-t}}{t} dt
\]

we study the 2D-motion of a particle of monolayer in plane moving in such a way, that the square of its speed decreases proportionally to the reciprocal value of time passed from the beginning of the motion.

We denote by \((t)\) the coordinate on \(X = T\), by \((t, r, \varphi)\) fibered coordinates on \(Y = T \times R^2\), and by \((t, r, \varphi, \dot{r}, \dot{\varphi})\) the associated coordinates on \(J^1(T, R^2)\).

This particle of monolayer governed by the jet Lagrangian function \(L: J^1(T, R^2) \to R\) defined by

\[
L(t, r, \dot{r}, \dot{\varphi}) = \frac{m}{2} \dot{r}^2 + \frac{mr^2}{2} \dot{\varphi}^2 - \frac{pr^5}{r} |V| \frac{2|\dot{r}|}{|\dot{\varphi}|} \cdot \frac{\dot{r}^{-1}}{u_{s(t,r)}} + U(t, r),
\]

(1)
where we have the following physical meanings: \( m \) is the mass of the particle; \( V \) is the LB-monolayer compressing speed; \( p \) is a constant monolayer parameter given by the physical formula:
\[
p = \frac{\pi^2 q^2 \rho_0^2}{\varepsilon \varepsilon_0 R_0^2};
\]

\( U_s(t, r) \) is an electro capillarity potential energy including the monomolecular layer function:
\[
U(t, r) = p \left\{ \frac{4}{3} r^5 + \frac{16}{15} (|V|t)r^4 + \frac{1}{30} (|V|t)^2 r^3 + \frac{1}{45} (|V|t)^3 r^2 + \frac{1}{45} (|V|t)^4 r \right. \\
+ \frac{2}{45} (|V|t)^5 \right. e^{\frac{2|V|t}{r}} - \frac{4}{45} \frac{(|V|t)^6}{r} f \left( \frac{2|V|t}{r} \right). \]

According to the Lagrange theory on fibered manifolds [3], to every Lagrangian there exists a unique Lepagean equivalent, denoted by \( \theta_\lambda \), where \( \lambda \) is a Lagrangian on \( J^1(T, R^2) \). The Euler-Lagrange form defined by the Lepagean equivalent depends only on the Lagrangian \( \lambda \). Therefore it is referred to as the Euler-Lagrange form of the Lagrangian \( \lambda \), and is denoted by \( E_\lambda \). Now, we shall find chart expressions for the Lepagean and the Euler-Lagrange forms.

In fibered coordinates \( (t, r, \varphi, r', \varphi') \) on \( f^1(T, R^2) \), the module of contact 1-forms (denoted by \( \Omega^1 J^1(T, R^2) \)) is locally generated by the forms \( \omega^1 = dr - r'dt \) and \( \omega^2 = d\varphi - \varphi'dt \). Put \( \lambda = L \, dt \) and denote
\[
\theta_\lambda = L \, dt + \frac{\partial L}{\partial r} \, \omega^1 + \frac{\partial L}{\partial \varphi} \, \omega^2, \tag{2}
\]
accordingly
\[
\theta_\lambda = \left( \frac{1}{2} \, m \, \left( \dot{r}^2 + r^2 \, \dot{\varphi}^2 \right) - p \, r^5 \, |V|e^{\frac{2|V|t}{r}} \frac{r^{-1}}{r} + U(t, r) \right) \, dt \\
+ \left( m \, \dot{r} + p \, r^5 \, |V|e^{\frac{2|V|t}{r}} \frac{r^{-2}}{r} \right) \omega^1 + (m \, r^2 \, \dot{\varphi}) \omega^2. \tag{3}
\]

We define a first order mechanical system \([\alpha] \) on the fibered manifold \( J^1(T, R^2) \to R \times R^2 \) represented by the 2-form with respect to (15)
\[
\alpha = d\theta_\lambda + F = \left( m r \dot{\varphi}^2 - \frac{5p \, r^4 \, |V|e^{\frac{2|V|t}{r}}}{r} + \frac{2pr^3 \, |V|^2 \, r \, e^{\frac{2|V|t}{r}}}{r} + \frac{\partial U}{\partial r} \right) dr \wedge dt \\
+ \left( \frac{2pr^4 \, |V|^2 \, e^{\frac{2|V|t}{r}}}{r^2} \right) dt \wedge \omega^1 + \left( \frac{5pr^4 \, |V|^2 \, e^{\frac{2|V|t}{r}}}{r^2} - \frac{2pr^3 \, |V|^2 \, te^{\frac{2|V|t}{r}}}{r^2} \right) dr \wedge \omega^1 \\
+ \left( m - \frac{2pr^5 \, |V|e^{\frac{2|V|t}{r}}}{r^3} \right) \, dr \wedge \omega^1 + (2mr \, \dot{\varphi}) dr \wedge \omega^2 + (mr^2 \, d\varphi \wedge \omega^2 + F. \tag{4}
\]
This mechanical system is related to the dynamical form with respect to (11)

\[ E = E_1 \, dr \wedge dt + E_2 \, d\varphi \wedge dt, \]

where

\[ E_1 = \left( m r \dot{\varphi} - \frac{10pr^4 |V| e^{2|V|t}}{\dot{r}} + \frac{4pr^3 r^3 |V|^2 e^{2|V|t}}{\dot{r}} - \frac{2pr^4 |V|^2 e^{2|V|t}}{\dot{r}^2} + \frac{\partial U}{\partial r} \right) - \left( m + \frac{2pr^5 |V|e^{-2|V|t}}{\dot{r}^2} \right) r, \]

and

\[ E_2 = -(2mr^2 + mr^2 \varphi). \]

We consider the nonholonomic constraint \( Q \) given by the equation

\[ \left( \frac{\partial f}{\partial r}, \frac{\partial f}{\partial \varphi} \right) = \text{rank}(2\dot{r},2\dot{\varphi}) = 1. \]

which means that the particle's speed decreases proportionally to \( \frac{1}{\sqrt{t}} \). In a neighborhood of the submanifold \( Q \)

\[ \text{rank} \left( \frac{\partial f}{\partial r}, \frac{\partial f}{\partial \varphi} \right) = \text{rank}(2\dot{r},2\dot{\varphi}) = 1. \]

Let \( U \subset J^1(T,R^2) \) be the set of all points, where \( \dot{\varphi} > 0 \), and consider on \( U \) the adapted coordinates \((t,r,\varphi,\dot{r},\dot{\varphi})\), where \( \ddot{\varphi} = \dot{\varphi} - g, \ g = \frac{1}{\sqrt{t}} - (\dot{r})^2 \) is the equation of the constraint (6) in normal form.

The constrained system \([\alpha_Q]\) related to the mechanical system \([\alpha]\) and the constraint \( Q \) is the equivalence class of the 2-form (with respect to (18))

\[ \alpha_Q = A'_1 \omega_1 \wedge dt + B'_{1,1} \omega_1 \wedge d\dot{r} + \Phi + \Phi(2) \]

on \( Q \), where \( \Phi \) is any 2-contact 2-form and \( \Phi(2) \) is any constraint 2-form defined on \( Q \).

Calculating \( L = L \circ \tau \) and calculating \( A'_1, B'_{1,1} \) by relationships (19), (20)

\[ L(t,r,\varphi,\dot{r},\dot{\varphi}) = L\left(t,r,\varphi,\dot{r},\sqrt{\frac{1}{t} - (\dot{r})^2}\right). \]

Then

\[ L = \frac{m}{2} r^2 + \frac{mr^2}{2} \left( \frac{1}{t} - \dot{r}^2 \right) - \frac{pr^5 |V| e^{2|V|t}}{\dot{r}} \cdot r^{-1} + U(t,r), \]

and

\[ A'_1 = mr g^2 + 2mr^2 - \frac{m\dot{r}^2}{2t} g^2 - \frac{10pr^4 t |V|^2 e^{-2|V|t}}{\dot{r}} + \frac{4pr^3 r^3 |V|^2 e^{-2|V|t}}{\dot{r}} - \frac{2pr^4 |V|^2 e^{-2|V|t}}{\dot{r}^2} + \frac{\partial U}{\partial r}, \]
A nonholonomic mechanical structure for the two-dimensional monolayer system

\[ B_{1,1} = m(r^2 - 1) - \frac{mr^2}{t\sigma^2} + 2pr^5 t \left| \frac{\sigma}{r} \right| \frac{2\sigma}{r^3}. \]

Reduced equation of motion of the constrained system is as follows

\[ [A_1 + B_{1,1} r] \circ J^2 \gamma = 0, \]

where \( \gamma = (t, r(t), \varphi(t)) \) is a section satisfying the constraint equation \( f \circ J^1 \gamma = 0. \)

**Lagrangian systems on fibered manifolds**

Throughout this section we consider a fibered manifold \( \pi: Y \to X \) with a one dimensional base space \( X \) and \( (m + 1) \)-dimensional total space \( Y \). We use jet prolongations \( \pi_1 : J^1(X, Y) \to X \) and \( \pi_2 : J^2(X, Y) \to X \) and jet projections \( \pi_{1,0} : J^1(X, Y) \to Y \) and \( \pi_{2,1} : J^2(X, Y) \to J^1(X, Y) \). Configuration space at a fixed time is represented by a fiber of the fibered manifold \( \pi \) and a corresponding phase space is then a fiber of the fibered manifold \( \pi_1 \). Local fibered coordinates on \( Y \) are denoted by \( (t, q^\sigma) \), where \( 1 \leq \sigma \leq m \). The associated coordinates on \( J^1(X, Y) \) and \( J^2(X, Y) \) are denoted by \( (t, q^\sigma, q^\sigma) \) and \( (t, q^\sigma, \dot{q}^\sigma, q^\sigma) \), respectively. In calculations we use either a canonical basis of one forms on \( J^1(X, Y) \), \( (dt, dq^\sigma, d\dot{q}^\sigma) \), or a basis adapted to the contact structure

\[ (dt, \omega^\sigma, d\dot{q}^\sigma), \]

where

\[ \omega^\sigma = dq^\sigma - \dot{q}^\sigma dt, \quad 1 \leq \sigma \leq m. \]

Whenever possible, the summation convention is used. If \( f(t, q^\sigma, \dot{q}^\sigma) \) is a function defined on an open set of \( J^1(X, Y) \) we write

\[ \frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial q^\sigma} \dot{q}^\sigma + \frac{\partial f}{\partial \dot{q}^\sigma} \ddot{q}^\sigma, \]

A differential form \( \rho \) is called contact if \( J^1 Y \circ \rho = 0 \) for every section \( \gamma \) of \( \pi \). A contact 2-form \( \rho \) is called 1-contact if for every vertical vector field \( \xi, i_\xi \rho_1 \) is a horizontal; \( \rho \) is 2-contact if \( i_\xi \rho_1 \) is 1-contact. The operator assigning to \( \rho \) it’s 1-contact part is denoted by \( p_1 \).

If \( \lambda \) is a Lagrangian on \( J^1(X, Y) \), we denote by \( \theta_\lambda \) its Lepage equivalent or Cartan form and \( E_\lambda \) its Euler-Lagrange form, respectively. Recall that \( E_\lambda = p_1 d\theta_\lambda \). In fibered coordinates where \( \lambda = L(t, q^\sigma, \dot{q}^\sigma) \) \( dt \), we have

\[ \theta_\lambda = L \, dt + \frac{\partial L}{\partial q^\sigma} \omega^\sigma, \]

and
where the components \( E_\sigma (L) = \frac{\partial L}{\partial \dot{q}^\sigma} - \frac{d}{dt} \frac{\partial L}{\partial q^\sigma} \), are the Euler-Lagrange expressions. Since the functions \( E_\sigma \) are affine in the second derivatives we write
\[
E_\sigma = A_\sigma + B_\sigma \dot{q}^\nu,
\]
where
\[
A_\sigma = \frac{\partial L}{\partial q^\sigma} - \frac{\partial^2 L}{\partial t \partial q^\sigma} - \frac{\partial^2 L}{\partial q^\nu \partial q^\sigma} \ddot{q}^\nu, \quad B_\sigma \nu = - \frac{\partial^2 L}{\partial q^\sigma \partial \dot{q}^\nu}.
\]

A section \( \gamma \) of \( \pi \) is called a path of the Euler-Lagrange form \( E_\lambda \) if
\[
E_\lambda \circ J^2 \gamma = 0.
\]
In fibered coordinates this equation represents a system of \( m \) second-order ordinary differential equations
\[
A_\sigma \left( t, \gamma^\nu, \frac{d\gamma^\nu}{dt} \right) + B_\sigma \beta \left( t, \gamma^\nu, \frac{d\gamma^\nu}{dt} \right) \frac{d^2 \gamma^\beta}{dt^2} = 0,
\]
for components \( \gamma^\nu(t) \) of a section \( \gamma \), where \( 1 \leq \nu \leq m \). These equations are called Euler-Lagrange equations or motion equations and their solutions are called paths.

Euler-Lagrange equations (14) can be written in the form
\[
J^1 \gamma^* \iota_\xi \alpha = 0,
\]
where \( \alpha = d\theta_\lambda + F \) is any 2-form defined on an open subset \( W \subset J^1(X,Y) \), such that \( p_\lambda \alpha = E_\lambda \), and \( F \) is a 2-contact 2-form. In fibered coordinates we have \( F = F_{\sigma\nu} \omega^\sigma \wedge \omega^\nu \), where \( F_{\sigma\nu}(t, q^\rho, \dot{q}^\rho) \) are arbitrary functions. Recall from [3] that the family of all such (local) 2-forms:
\[
\alpha = d\theta_\lambda + F = A_\sigma \omega^\sigma \wedge dt + B_{\sigma\nu} \omega^\sigma \wedge dq^\nu + F
\]
is called a first order Lagrangian system, and is denoted by \( [\alpha] \).

A non-holonomic constrained mechanical system is defined on the \((2m+1-k)\)-dimensional constraint submanifold \( Q \subset J^1(X,Y) \) fibered over \( Y \) and given by \( k \) equations
\[
f^i(t, q^1, \ldots, q^m, \dot{q}^1, \ldots, \dot{q}^m) = 0, \quad 1 \leq i \leq k,
\]
where
\[
\text{rank} \left( \frac{\partial f^i}{\partial q^\sigma} \right) = k,
\]
or following [6], equivalently in an explicit form
\[
\dot{q}^{m-k+i} = g^i(t, q^\sigma, \dot{q}^1, \dot{q}^2, \ldots, \dot{q}^{m-k}), \quad 1 \leq i \leq k.
\]
By a nonholonomic constrained system arising from the Lagrangian system $[\alpha] \text{ and constraint forms on the constraint submanifold } Q$, we mean the equivalence class $[\alpha_Q]$ on $Q$, where

$$\alpha_Q = \iota^* d\theta^\lambda + \tilde{F} + \phi_{(2)},$$

where $\tilde{F}$ is any 2-contact $\pi_{1,0}$ horizontal 2-form and $\phi_2$ is any constraint 2-form defined on $Q$, and $\iota$ is the canonical embedding of $Q$ into $J^1(X,Y)$. The local form of $[\alpha_Q]$ is

$$\alpha_Q = \sum_{l=1}^{m-k} A'_l \omega^l \land dt + \sum_{l_2=1}^{m-k} B'_{l_2} \omega^l \land dq^s + \tilde{F} + \phi_{(2)} ,$$

where the components $A'_l$ and $B'_{l_2}$ are given by

$$A'_l = \frac{\partial L}{\partial \dot{q}^l} + \frac{\partial L}{\partial q^{m-k+i}} \frac{\partial g^i}{\partial q^l} - \frac{\partial c}{\partial t} \frac{\partial L}{\partial q^l} - \left( \frac{\partial L}{\partial q^{m-k+i}} \right)_t \frac{\partial}{\partial q^l} \left( \frac{\partial g^i}{\partial q^l} \right) - \frac{\partial g^j}{\partial q^{m-k+i}} \frac{\partial g^i}{\partial q^l} ,$$

$$B'_{l_2} = -\frac{\partial^2 L}{\partial q^l \partial q^s} + \left( \frac{\partial L}{\partial q^{m-k+i}} \right)_t \frac{\partial^2 g^i}{\partial q^l \partial q^s} .$$

$\tilde{L} = L \circ \iota$, and

$$\frac{\partial c}{\partial t} \dot{q}^s + g^i \frac{\partial}{\partial q^{m-k+i}} .$$

The equations of the motion of the constrained system $[\alpha_Q]$ in fibered coordinates take the form

$$\left( A'_l + \sum_{s=1}^{m-k} B'_{l_2} \ddot{q}^s \right) \circ J^2 \ddot{\gamma} = 0 ,$$

for components $y^1(t), y^2(t), \ldots, y^{m-k}(t)$ of a $Q -$ admissible section $\ddot{\gamma}$ dependent on time $t$ and parameters $q^{m-k+1}, q^{m-k+2}, \ldots, q^m$, which have to be determined as functions $y^{m-k+1}(t), y^{m-k+2}(t), \ldots, y^m(t)$ from the equations (17) of the constraint

$$\frac{dq^{m-k+i}}{dt} = g^i \left( t, q^\sigma, \frac{dq^1}{dt}, \frac{dq^2}{dt}, \ldots, \frac{dq^{m-k}}{dt} \right), \quad 1 \leq i \leq k.$$
References


