

## A Nonholonomic Mechanical Structure for the Two-Dimensional Monolayer Systems

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### Abstract

In this paper we study the dynamics of the 2D-motion of a particle of monolayer. A geometric approach to nonholonomic constrained mechanical systems is applied to a problem from the two dimensional geometric dynamics of the Langmuir-Blodgett monolayer. We consider a constraint submanifold of the 1-jet space  $J^1(T, R^2)$ , corresponding to the given constraint condition in a monolayer space and construct the corresponding constrained mechanical system on the constraint submanifold. Then the equation of motion defined on the constraint submanifold is presented.

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### Introduction

Physical systems are often subjected to various types of constraints. Generally, these constraints are position, or geometric (holonomic), constraints, or velocity (nonholonomic) constraints. If a system has constraint equations that involve velocities or derivatives of system coordinates, the constraint equations are said to be nonholonomic and the mechanical system is said to be a nonholonomic system. Although almost all the work on nonholonomic systems is concerned with the case of constraints linear in components of velocities, but a geometric theory covering general nonholonomic systems has been developed by Krupkova [2], [3].

An example of such a system might be monolayer systems. In this paper we present geometric concept of the theory of nonholonomic mechanical systems developed by

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Krupkova [3] and consider the application of this theory to a problem from the two dimensional geometric dynamics of the Langmuir-Blodgett monolayer [4]. To this aim, we consider a constraint submanifold of the 1-jet space  $J^1(T, R^2)$  corresponding to the given constraint conditions and construct the corresponding constrained mechanical system on the constraint submanifold. Then the equations of motion defined on the constraint submanifold are presented.

Section 3 of this paper contains a brief outline of basic geometrical concepts of nonholonomic mechanical systems that we will need. For proofs and more details see [2], [3].

### Nonholonomic mechanical systems in a monolayer space

In this section we begin with a brief introduction to a monolayer space [1], [5] and then describe the nonholonomic mechanical system related to it. To obtain the 2D-motion equation of a particle of monolayer, we define a first order mechanical system  $[\alpha]$  in this space and calculate the nonholonomic constrained system  $[\alpha_Q]$  related to the mechanical system  $[\alpha]$ .

We start with the usual physical time defined by the Euclidian manifold  $(T = [0, \infty))$  and we also consider the plane manifold  $R^2$  having the polar coordinates  $(r, \varphi)$ , where  $r > 0$  and  $\varphi \in [0, 2\pi)$ , and construct the 1-jet vector bundle  $J^1(T, R^2) \rightarrow R \times R^2$ , locally endowed with the coordinates  $(t, q^1, q^2, \dot{q}^1, \dot{q}^2) := (t, r, \varphi, \dot{r}, \dot{\varphi})$ .

Using the special function:

$$f(z) = - \int_{-z}^{\infty} \frac{e^{-t}}{t} dt$$

we study the 2D-motion of a particle of monolayer in plane moving in such a way, that the square of its speed decreases proportionally to the reciprocal value of time passed from the beginning of the motion.

We denote by  $(t)$  the coordinate on  $X = T$ , by  $(t, r, \varphi)$  fibered coordinates on  $Y = T \times R^2$ , and by  $(t, r, \varphi, \dot{r}, \dot{\varphi})$  the associated coordinates on  $J^1(T, R^2)$ .

This particle of monolayer governed by the jet Lagrangian function  $L: J^1(T, R^2) \rightarrow R$  defined by

$$L(t, r, \dot{r}, \dot{\varphi}) = \frac{m}{2} \dot{r}^2 + \frac{mr^2}{2} \dot{\varphi}^2 - \underbrace{pr^5 |V| e^{\frac{2|V|t}{r}} \cdot \dot{r}^{-1}}_{U_s(t, r)} + U(t, r), \quad (1)$$

where we have the following physical meanings:  $m$  is the *mass* of the particle;  $V$  is the *LB-monolayer compressing speed*;  $p$  is a *constant monolayer parameter* given by the physical formula:

$$p = \frac{\pi^2 q^2}{\varepsilon \varepsilon_0} \frac{\rho_0^2}{R_0^2};$$

$U_s(t, r)$  is an *electro capillarity potential energy* including the monomolecular layer function:

$$U(t, r) = p \left\{ \left[ -\frac{4}{3} r^5 + \frac{16}{15} (|V|t) r^4 + \frac{1}{30} (|V|t)^2 r^3 + \frac{1}{45} (|V|t)^3 r^2 + \frac{1}{45} (|V|t)^4 r + \frac{2}{45} (|V|t)^5 \right] e^{\frac{2|V|t}{r}} - \frac{4}{45} + \frac{(|V|t)^6}{r} f\left(\frac{2|V|t}{r}\right) \right\}$$

According to the Lagrange theory on fibered manifolds [3], to every Lagrangian there exists a unique Lepagean equivalent, denoted by  $\theta_\lambda$ , where  $\lambda$  is a Lagrangian on  $J^1(T, R^2)$ . The Euler-Lagrange form defined by the Lepagean equivalent depends only on the Lagrangian  $\lambda$ . Therefore it is referred to as the Euler-Lagrange form of the Lagrangian  $\lambda$ , and is denoted by  $E_\lambda$ . Now, we shall find chart expressions for the Lepagean and the Euler-Lagrange forms.

In fibered coordinates  $(t, r, \varphi, \dot{r}, \dot{\varphi})$  on  $J^1(T, R^2)$ , the module of contact 1-forms (denoted by  $\Omega^1 J^1(T, R^2)$ ) is locally generated by the forms  $\omega^1 = dr - \dot{r} dt$  and  $\omega^2 = d\varphi - \dot{\varphi} dt$ . Put  $\lambda = L dt$  and denote

$$\theta_\lambda = L dt + \frac{\partial L}{\partial \dot{r}} \omega^1 + \frac{\partial L}{\partial \dot{\varphi}} \omega^2, \quad (2)$$

accordingly

$$\begin{aligned} \theta_\lambda = & \left( \frac{1}{2} m(\dot{r}^2 + r^2 \dot{\varphi}^2) - p r^5 |V| e^{\frac{2|V|t}{r}} \dot{r}^{-1} + U(t, r) \right) dt \\ & + \left( m \dot{r} + p r^5 |V| e^{\frac{2|V|t}{r}} \dot{r}^{-2} \right) \omega^1 + (m r^2 \dot{\varphi}) \omega^2. \end{aligned} \quad (3)$$

We define a first order mechanical system  $[\alpha]$  on the fibered manifold  $J^1(T, R^2) \rightarrow R \times R^2$  represented by the 2-form with respect to (15)

$$\begin{aligned} \alpha = d\theta_\lambda + F = & \left( m r \dot{\varphi}^2 - \frac{5p r^4 |V| e^{\frac{2|V|t}{r}}}{\dot{r}} + \frac{2p r^3 |V|^2 t e^{\frac{2|V|t}{r}}}{\dot{r}} + \frac{\partial U}{\partial r} \right) dr \wedge dt \\ & + \left( \frac{2p r^4 |V|^2 e^{\frac{2|V|t}{r}}}{\dot{r}^2} \right) dt \wedge \omega^1 + \left( \frac{5p r^4 |V| e^{\frac{2|V|t}{r}}}{\dot{r}^2} - \frac{2p r^3 |V|^2 t e^{\frac{2|V|t}{r}}}{\dot{r}^2} \right) dr \wedge \omega^1 \\ & + \left( m - \frac{2p r^5 |V| e^{\frac{2|V|t}{r}}}{\dot{r}^3} \right) d\dot{r} \wedge \omega^1 + (2m r \dot{\varphi}) dr \wedge \omega^2 + (m r^2) d\dot{\varphi} \wedge \omega^2 + F. \end{aligned} \quad (4)$$

This mechanical system is related to the dynamical form with respect to (11)

$$E = E_1 dr \wedge dt + E_2 d\varphi \wedge dt, \quad (5)$$

where

$$E_1 = \left( mr\dot{\varphi} - \frac{10pr^4 |V| e^{\frac{2|V|t}{r}}}{\dot{r}} + \frac{4pt r^3 |V|^2 e^{\frac{2|V|t}{r}}}{\dot{r}} - \frac{2pr^4 |V|^2 e^{\frac{2|V|t}{r}}}{\dot{r}^2} + \frac{\partial U}{\partial r} \right) - \left( m + \frac{2pr^5 |V| e^{\frac{2|V|t}{r}}}{\dot{r}^2} \right) r'',$$

and

$$E_2 = -(2mrr' + mr^2\varphi).$$

We consider the nonholonomic constraint  $Q$  given by the equation

$$f(t, r, \varphi, r', \varphi') \equiv [(r')^2 + (\varphi')^2] - \frac{1}{t} = 0, \quad (6)$$

which means that the particle's speed decreases proportionally to  $\frac{1}{\sqrt{t}}$ . In a neighborhood of the submanifold  $Q$

$$\text{rank} \left( \frac{\partial f}{\partial r'}, \frac{\partial f}{\partial \varphi'} \right) = \text{rank} (2\dot{r}, 2\dot{\varphi}) = 1. \quad (7)$$

Let  $U \subset J^1(T, R^2)$  be the set of all points, where  $\dot{\varphi} > 0$ , and consider on  $U$  the adapted coordinates  $(t, r, \varphi, r', \bar{f})$ , where  $\bar{f} = \dot{\varphi} - g$ ,  $g = \sqrt{\frac{1}{t} - (\dot{r})^2}$  is the equation of the constraint (6) in normal form.

The constrained system  $[\alpha_Q]$  related to the mechanical system  $[\alpha]$  and the constraint  $Q$  is the equivalence class of the 2-form (with respect to (18))

$$\alpha_Q = A'_1 \omega^1 \wedge dt + B'_{1,1} \omega^1 \wedge d\dot{r} + \bar{F} + \phi_{(2)} \quad (8)$$

on  $Q$ , where  $\bar{F}$  is any 2-contact 2-form and  $\phi_{(2)}$  is any constraint 2-form defined on  $Q$ .

Calculating  $\bar{L} = L \circ \iota$  and calculating  $A'_1, B'_{1,1}$  by relationships (19), (20)

$$\bar{L}(t, r, \varphi, r', \dot{\varphi}) = L \left( t, r, \varphi, r', \sqrt{\frac{1}{t} - (\dot{r})^2} \right).$$

Then

$$\bar{L} = \frac{m}{2} \dot{r}^2 + \frac{mr^2}{2} \left( \frac{1}{t} - \dot{r}^2 \right) - \underbrace{pr^5 |V| e^{\frac{2|V|t}{r}} \cdot \dot{r}^{-1}}_{U_{S(t,r)}} + U(t, r),$$

and

$$A'_1 = mr g^2 + 2mrr' - \frac{m\dot{r}r^2}{2t^2} g^2 - \frac{10pr^4 t |V|^2 e^{\frac{2|V|t}{r}}}{\dot{r}} + \frac{4pr^3 t |V|^2 e^{\frac{2|V|t}{r}}}{\dot{r}} - \frac{2pr^4 |V|^2 e^{\frac{2|V|t}{r}}}{\dot{r}^2} + \frac{\partial U}{\partial r},$$

$$B'_{1,1} = m(r^2 - 1) - \frac{mr^2}{tg^2} + \frac{2pr^5 t|V|e^{\frac{2|V|t}{r}}}{r^3}.$$

Reduced equation of motion of the constrained system is as follows

$$[A'_1 + B'_{1,1} r] \circ J^2 \bar{\gamma} = 0, \quad (9)$$

where  $\bar{\gamma} = (t, r(t), \varphi(t))$  is a section satisfying the constraint equation  $f \circ J^1 \gamma = 0$ .

### Lagrangian systems on fibered manifolds

Throughout this section we consider a fibered manifold  $\pi: Y \rightarrow X$  with a one dimensional base space  $X$  and  $(m+1)$ -dimensional total space  $Y$ . We use jet prolongations  $\pi_1: J^1(X, Y) \rightarrow X$  and  $\pi_2: J^2(X, Y) \rightarrow X$  and jet projections  $\pi_{1,0}: J^1(X, Y) \rightarrow Y$  and  $\pi_{2,1}: J^2(X, Y) \rightarrow J^1(X, Y)$ . Configuration space at a fixed time is represented by a fiber of the fibered manifold  $\pi$  and a corresponding phase space is then a fiber of the fibered manifold  $\pi_1$ . Local fibered coordinates on  $Y$  are denoted by  $(t, q^\sigma)$ , where  $1 \leq \sigma \leq m$ . The associated coordinates on  $J^1(X, Y)$  and  $J^2(X, Y)$  are denoted by  $(t, q^\sigma, \dot{q}^\sigma)$  and  $(t, q^\sigma, \dot{q}^\sigma, \ddot{q}^\sigma)$ , respectively. In calculations we use either a canonical basis of one forms on  $J^1(X, Y)$ ,  $(dt, dq^\sigma, d\dot{q}^\sigma)$ , or a basis adapted to the contact structure

$$(dt, \omega^\sigma, d\dot{q}^\sigma),$$

where

$$\omega^\sigma = dq^\sigma - \dot{q}^\sigma dt, \quad 1 \leq \sigma \leq m.$$

Whenever possible, the summation convention is used. If  $f(t, q^\sigma, \dot{q}^\sigma)$  is a function defined on an open set of  $J^1(X, Y)$  we write

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial q^\sigma} \dot{q}^\sigma + \frac{\partial f}{\partial \dot{q}^\sigma} q^\sigma, \quad \frac{\bar{d}f}{dt} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial q^\sigma} \dot{q}^\sigma.$$

A differential form  $\rho$  is called *contact* if  $J^1 \gamma^* \rho = 0$  for every section  $\gamma$  of  $\pi$ . A contact 2-form  $\rho$  is called *1-contact* if for every vertical vector field  $\xi$ ,  $i_\xi \rho_1$  is a horizontal;  $\rho$  is 2-contact if  $i_\xi \rho_1$  is 1-contact. The operator assigning to  $\rho$  its 1-contact part is denoted by  $p_1$ .

If  $\lambda$  is a Lagrangian on  $J^1(X, Y)$ , we denote by  $\theta_\lambda$  its *Lepage equivalent* or *Cartan form* and  $E_\lambda$  its Euler-Lagrange form, respectively. Recall that  $E_\lambda = p_1 d\theta_\lambda$ . In fibered coordinates where  $\lambda = L(t, q^\sigma, \dot{q}^\sigma) dt$ , we have

$$\theta_\lambda = L dt + \frac{\partial L}{\partial \dot{q}^\sigma} \omega^\sigma, \quad (10)$$

and

$$E_\lambda = E_\sigma (L) \omega^\sigma \wedge dt, \quad (11)$$

where the components  $E_\sigma (L) = \frac{\partial L}{\partial q^\sigma} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}^\sigma}$ , are the *Euler-Lagrange expressions*.

Since the functions  $E_\sigma$  are affine in the second derivatives we write

$$E_\sigma = A_\sigma + B_{\sigma\nu} \dot{q}^\nu,$$

where

$$A_\sigma = \frac{\partial L}{\partial q^\sigma} - \frac{\partial^2 L}{\partial t \partial \dot{q}^\sigma} - \frac{\partial^2 L}{\partial q^\nu \partial \dot{q}^\sigma} \dot{q}^\nu, \quad B_{\sigma\nu} = -\frac{\partial^2 L}{\partial \dot{q}^\sigma \partial \dot{q}^\nu}. \quad (12)$$

A section  $\gamma$  of  $\pi$  is called a *path* of the Euler-Lagrange form  $E_\lambda$  if

$$E_\lambda \circ J^2 \gamma = 0. \quad (13)$$

In fibered coordinates this equation represents a system of  $m$  second-order ordinary differential equations

$$A_\sigma \left( t, \gamma^\nu, \frac{d\gamma^\nu}{dt} \right) + B_{\sigma\rho} \left( t, \gamma^\nu, \frac{d\gamma^\nu}{dt} \right) \frac{d^2 \gamma^\rho}{dt^2} = 0, \quad (14)$$

for components  $\gamma^\nu(t)$  of a section  $\gamma$ , where  $1 \leq \nu \leq m$ . These equations are called *Euler-Lagrange equations* or *motion equations* and their solutions are called *paths*.

Euler-Lagrange equations (14) can be written in the form

$$J^1 \gamma^* i_\xi \alpha = 0,$$

where  $\alpha = d\theta_\lambda + F$  is any 2-form defined on an open subset  $W \subset J^1(X, Y)$ , such that  $p_1 \alpha = E_\lambda$ , and  $F$  is a 2-contact 2-form. In fibered coordinates we have  $F = F_{\sigma\nu} \omega^\sigma \wedge \omega^\nu$ , where  $F_{\sigma\nu}(t, q^\rho, \dot{q}^\rho)$  are arbitrary functions. Recall from [3] that the family of all such (local) 2-forms:

$$\alpha = d\theta_\lambda + F = A_\sigma \omega^\sigma \wedge dt + B_{\sigma\nu} \omega^\sigma \wedge dq^\nu + F \quad (15)$$

is called a *first order Lagrangian system*, and is denoted by  $[\alpha]$ .

A non-holonomic constrained mechanical system is defined on the  $(2m+1-k)$ -dimensional constraint submanifold  $Q \subset J^1(X, Y)$  fibered over  $Y$  and given by  $k$  equations

$$f^i(t, q^1, \dots, q^m, \dot{q}^1, \dots, \dot{q}^m) = 0, \quad 1 \leq i \leq k,$$

where

$$\text{rank} \left( \frac{\partial f^i}{\partial \dot{q}^\sigma} \right) = k, \quad (16)$$

or following [6], equivalently in an explicit form

$$\dot{q}^{m-k+i} = g^i(t, q^\sigma, \dot{q}^1, \dot{q}^2, \dots, \dot{q}^{m-k}), \quad 1 \leq i \leq k. \quad (17)$$

By a nonholonomic constrained system arising from the Lagrangian system  $[\alpha]$  and constraint forms on the constraint submanifold  $Q$ , we mean the equivalence class  $[\alpha_Q]$  on  $Q$ , where

$$\alpha_Q = \iota^* d\theta_\lambda + \bar{F} + \phi_{(2)},$$

where  $\bar{F}$  is any 2-contact  $\pi_{1,0}$  horizontal 2-form and  $\phi_2$  is any constraint 2-form defined on  $Q$ , and  $\iota$  is the canonical embedding of  $Q$  into  $J^1(X, Y)$ . The local form of  $[\alpha_Q]$  is

$$\alpha_Q = \sum_{l=1}^{m-k} A'_l \omega^l \wedge dt + \sum_{l,s=1}^{m-k} B'_{ls} \omega^l \wedge d\dot{q}^s + \bar{F} + \phi_{(2)}, \quad (18)$$

where the components  $A'_l$  and  $B'_{ls}$  are given by

$$A'_l = \frac{\partial \bar{L}}{\partial q^l} + \frac{\partial \bar{L}}{\partial q^{m-k+i}} \frac{\partial g^i}{\partial \dot{q}^l} - \frac{\bar{d}c}{dt} \frac{\partial \bar{L}}{\partial \dot{q}^l} + \left( \frac{\partial L}{\partial \dot{q}^{m-k+j}} \right)_l \left[ \frac{\bar{d}c}{dt} \left( \frac{\partial g^j}{\partial \dot{q}^l} \right) - \frac{\partial g^j}{\partial q^l} - \frac{\partial g^j}{\partial q^{m-k+i}} \frac{\partial g^i}{\partial \dot{q}^l} \right],$$

$$B'_{ls} = -\frac{\partial^2 \bar{L}}{\partial \dot{q}^l \partial \dot{q}^s} + \left( \frac{\partial L}{\partial \dot{q}^{m-k+i}} \right)_l \frac{\partial^2 g^i}{\partial \dot{q}^l \partial \dot{q}^s},$$

$$\bar{L} = L \circ \iota, \text{ and}$$

$$\frac{\bar{d}c}{dt} = \frac{\partial}{\partial t} + \dot{q}^s \frac{\partial}{\partial q^s} + g^i \frac{\partial}{\partial q^{m-k+i}}.$$

The equations of the motion of the constrained system  $[\alpha_Q]$  in fibered coordinates take the form

$$\left( A'_l + \sum_{s=1}^{m-k} B'_{ls} \ddot{q}^s \right) \circ J^2 \bar{\gamma} = 0. \quad (19)$$

for components  $\gamma^1(t), \gamma^2(t), \dots, \gamma^{m-k}(t)$  of a  $Q$ -admissible section  $\bar{\gamma}$  dependent on time  $t$  and parameters  $q^{m-k+1}, q^{m-k+2}, \dots, q^m$ , which have to be determined as functions  $\gamma^{m-k+1}(t), \gamma^{m-k+2}(t), \dots, \gamma^m(t)$  from the equations (17) of the constraint

$$\frac{dq^{m-k+i}}{dt} = g^i \left( t, q^\sigma, \frac{dq^1}{dt}, \frac{dq^2}{dt}, \dots, \frac{dq^{m-k}}{dt} \right), \quad 1 \leq i \leq k.$$

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