# Amenability and Weak Amenability of the Semigroup Algebra $\ell^1(S_T)$

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#### **Abstract**

Let **S** be a semigroup with a left multiplier T on **S**. A new product on **S** is defined by T related to **S** and T such that **S** and the new semigroup  $\mathbf{S}_T$  have the same underlying set as **S**. It is shown that if T is injective then  $\ell^1(\mathbf{S}_T) \cong \ell^1(\mathbf{S})_{\widetilde{T}}$  where,  $\widetilde{T}$  is the extension of T on  $\ell^1(\mathbf{S})$ . Also, we show that if T is bijective, then  $\ell^1(\mathbf{S})$  is amenable if and only if  $\ell^1(\mathbf{S}_T)$  is so. Moreover, if **S** completely regular, then  $\ell^1(\mathbf{S}_T)$  is weakly amenable.

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#### Introduction

Let S be a semigroup and T be a left multiplier on S. We present a general method of defining a new product on S which makes S a semigroup. Let  $S_T$  denote S with the new product. These two semigroups are sometims different and we try to find conditions on S and T such that the semigroups S and  $S_T$  have the same properties. This idea has started by Birtel in [1] for Banach algebras and continued by Larsen in [11]. Recently, this notion developed by some authors, for more details see [1], [10], [11], [12] and [15]. One of the best result in this work expresses that  $L^1(G)_T$  is Arens regular if and only if G is a compact group [10]. We continue this direction on the regularity of S and  $S_T$  and the amenability of their semigroup algebras.

The term of semigroup will be a non-empty set S endowed with an associative binary operation on S, defined by  $(s,t) \rightarrow st$ . If S is also a Hausdorff topological space and the binary operation is jointly continuous, then S is called a topological semigroup.

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Let  $p \in S$ . Then p is an idempotent if  $p^2 = p$ . The set of all idempotents of S is denoted by E(S).

An element e is a left (right) identity if es = s (resp. se = s) for all  $s \in S$ . An element  $e \in S$  is an identity if it is a left and a right identity. An element z is a left (resp. right) zero if zs = z (resp. sz = z) for all  $s \in S$ . An element  $z \in S$  is a zero if it is a left and a right zero. We denote any zero of S by  $0_S$  (or  $z_S$ ). An element  $p \in S$  is a regular element of S if there exists  $t \in S$  such that p = ptp and p is completely regular if it is regular and p = tp. We say that  $p \in S$  has an inverse if there exists  $t \in S$  such that p = ptp and p = tp. Note that the inverse of element  $p \in S$  need not be unique. If  $p \in S$  has an inverse, then p is regular and vise versa. Since, if  $p \in S$  is regular, there exists  $s \in S$  such that p = psp. Let p = tpsp. Let p = tpsp. Then

$$p = psp = (psp)sp = p(sps)p = ptp$$
,  $t = sps = s(psp)s = (sps)p(sps) = tpt$ .

So p has an inverse. We say that S is a regular (resp. completely regular) semigroup if each  $p \in S$  is regular (resp. completely regular). Also S is an inverse semigroup if each  $p \in S$  has a unique inverse. The map  $T : S \to S$  is called a left (resp. right) multiplier if

$$T(st) = T(s)t$$
 (resp.  $T(st) = sT(t)$ ) (s,  $t \in S$ ).

The map  $T: S \to S$  is a multiplier if it is a left and right multiplier.Let S be a topological semigroup. The net  $(e_{\alpha}) \subseteq S$  is a left (resp. right) approximate identity if  $\lim_{\alpha} e_{\alpha}t = t$ . (resp.  $\lim_{\alpha} t e_{\alpha} = t$ ) (teS). The net  $(e_{\alpha}) \subseteq S$  is an approximate identity if it is a left and a right approximate identity.

Let S be a discrete semigroup. We denote by  $\ell^1(S)$  the Banach space of all complex function  $f: S \to \mathbb{C}$  having the form

$$f = \sum_{s \in S} f(s) \delta_s$$
,

such that  $\sum_{s \in S} |f(s)| = ||f||_1$  is finite, where  $\delta_s$  is the point mass at  $\{s\}$ . For  $f, g \in \ell^1(S)$  we define the convolution product on  $\ell^1(S)$  as fallow:

$$f * g(s) = \sum_{t_1 t_2 = s} f(t_1)g(t_2)$$
 (seS),

with this product  $\ell^1(S)$  becomes a Banach algebra and is called the semigroup algebra on S.

Remark 1.1. If  $f \in \ell^1(S)$  then f = 0 on S except at most on a countable subset of S. In other words, the set  $A = \{s \in S: \ f(s) \neq 0\}$  is at most countable. Since, if  $A_n = \{s \in S: \ |f(s)| \geq \frac{1}{n}\}$ ,  $A = \bigcup_{n \in N} A_n$ . Set  $\|f\|_1 = M$  and  $n \in N$  is fixed. Then we have

$$M = \sum_{s \in S} |f(s)| \ge \sum_{s \in A_n} |f(s)| \ge \sum_{s \in A_n} \frac{1}{n} = \frac{1}{n} |A_n|,$$

where  $|A_n|$  is the cardinality of  $A_n$ . So  $|A_n| \le nM$ . Hence  $A_n$  is a finite subset of S and thus A is at most countable.

### Semigroup S<sub>T</sub>

Let  $T \in Mul_1(S)$ . Then we define a new binary operation " $\circ$ " on S as follow:

$$s \circ t = s T(t) \quad (s, t \in S).$$

The set S equipt with the new operation " $\circ$ " is denoted by  $S_T$  and sometimes called "induced semigroup of S". Now we have the following results.

**Theorem 2.1.** Let **S** be a Semigroup. Then (i) if  $T \in Mul_l(S)$  then  $S_T$  is a semigroup. The converse is true if **S** is left cancellative and T is surjective.

- (ii) If  $\mathbf{S}_T$  is left cancellative and T is surjective, then  $T^{-1} \in Mul_l(\mathbf{S})$ .
- (iii) If  $\bf S$  is a topological semigroup and  $\bf S_T$  has a left approximate identity then  $T^{-1} \in Mul_1(\bf S)$ .

**Proof.** i) Let  $T \in Mul_1(S)$  and take  $r, s, t \in S$ . Then

$$r \circ (s \circ t) = r T(s \circ t) = r T(s T(t)) = r T(s)T(t) = (r T(s)) T(t)$$
  
=  $(r \circ s) \circ t$ 

So,  $S_T$  is a semigroup.

Conversely, suppose that **S** is left cancellative and take r,s,t  $\epsilon$ **S**. Since T is surjective, there exists  $u\epsilon$ **S** such that T(u) = t. Then

$$rT(st) = rT(sT(u)) = r \circ (s \circ u) = (r \circ s) \circ u = (rT(s))T(u)$$
$$= r(T(s)t).$$

By the left cancellativity of **S**, we have T(st) = T(s)t  $(r, s \in S)$ . So, T is a left multiplier.

ii) We must prove that T is injective. To do this end, take  $r,s,u\in S$  and let T(r)=T(s). Then  $u\circ r=uT(r)=uT(s)=u\circ s$ . So r=s, since  $S_T$  is left cancellative. Hence  $T^{-1}$  exists.

Now, we show that  $T^{-1} \in Mul_1(S)$ . Take  $r, s \in S$ . Then

$$\begin{split} T^{-1}(rs) &= T^{-1}[TT^{-1}(r)s] = T^{-1}[T(T^{-1}(r)s)] \\ &= (T^{-1}T)[T^{-1}(r)s] = T^{-1}(r)s \,. \end{split}$$

iii) It is enough to show that T is injective. Take  $r, s \in S$  and suppose that T(r) = T(s). Then

$$r = \lim_{\alpha} e_{\alpha} \circ \ r \ = \lim_{\alpha} e_{\alpha} T(r) = \lim_{\alpha} e_{\alpha} T(s) = \lim_{\alpha} e_{\alpha} \circ \ s = s.$$

There are many properties that induced from **S** to semigroup  $S_T$ . But sometimes they are different.

**Theorem2.2.** Let **S** be a Hausdorff topological semigroup and  $T \in Mul_1(S)$ . If **S** is commutative then so is  $S_T$ . The converse is true if  $\overline{T(S)} = S$ .

**Proof.** Suppose **S** is commutative and take  $r, s \in S$ . Then

$$r \circ s = r T(s) = T(s)r = T(sr) = T(rs) = T(r) s = sT(r) = s \circ r$$
.

So,  $S_T$  is commutative.

Conversely, Let  $\mathbf{S}_T$  be commutative and take  $r, s \in \mathbf{S}$ . Then there exist nets  $(r_\alpha)$  and  $(s_\beta)$  in  $\mathbf{S}$  such that  $\lim_\alpha T(r_\alpha) = r$  and  $\lim_\beta T(s_\beta) = s$ .

So, we have

$$rs = \lim_{\alpha} \lim_{\beta} \mathbf{T} (r_{\alpha} \circ s_{\beta}) = \lim_{\alpha} \lim_{\beta} \mathbf{T} (s_{\beta} \circ r_{\alpha}) = \lim_{\alpha} \lim_{\beta} \mathbf{T} (s_{\beta}) \mathbf{T} (r_{\alpha}) = s r.$$

Thus **S** is commutative.

In the sequel, we investigate some relations between two semigroup S and  $S_T$  according to the role of the left multiplier T.

**Theorem 2.3.** Let **S** be a semigroup and  $T \in Mul_1(S)$ . Then

- (i) If T is surjective and  $S_T$  is an inverse semigroup then S is an inverse semigroup and  $T(s^{-1}) = T(s)^{-1}$  for all  $s \in S$ .
- (ii) If  $S_T$  is an inverse semigroup and T is injective then T(S) is an inverse subsemigroup of S.
- (iii) If T is bijective then  $S_T$  is an inverse semigroup if and only if S is an inverse semigroup.

**Proof.** i) Suppose that  $S_T$  is an inverse semigroup and T is surjective. Define the map  $\phi: S_T \to S$  by  $\phi(s) = T(s)$ . Take  $r, s \in S$ , then

$$\varphi(r \circ s) = T(r \circ s) = T(r)T(s) = \varphi(r)\varphi(s).$$

So,  $\varphi$  is an epimorphism from  $S_T$  onto S, since T is surjective. By theorem 5.1.4[7], S is an inverse semigroup and  $T(s^{-1}) = T(s)^{-1}$  for all  $s \in S$ .

ii) Suppose that T is injective and  $S_T$  is an inverse semigroup. Evidently, T(S) is a subsemigroup of S. We show that it is an inverse semigroup. Take  $s \in T(S)$ . There exists  $t \in S$  such that s = T(t). Also, there exists a unique element  $u \in S$  such that  $t = t_0 u_0 t$ , since  $S_T$  is an inverse semigroup. Therefore, T(t) = T(t)T(u)T(t), or  $s = s_0 T(u)_0 s$ . Of course, T(u) is unique because  $u \in S$  is unique and T is injective. Hence T(S) is an inverse subsemigroup of S.

iii) Suppose that T is bijective and let  $S_T$  be an inverse semigroup. Since T is injective and surjective, by (i) and (ii), S = T(S) is an inverse semigroup.

Conversely, suppose that S is an inverse semigroup. Since T is bijective, by theorem 2.1(ii),  $T^{-1} \in Mul_l(S)$ . So  $\varphi^{-1}$ :  $S \to S_T$  defined by  $\varphi^{-1}(s) = T^{-1}(s)$  is an epimorphism. Hence by (i)  $S_T$  is an inverse semigroup

We say that  $T \in Mul_l(S)$  is an inner left multiplier if it has the form  $T = L_s$  for some  $s \in S$  where  $L_s(t) = s t$   $(t \in S)$ .

If  $T \in Mul_l(S)$  is inner, then each ideal of S is permanent under T; that is  $T(I) \subseteq I$  for all ideal I of S. It is easily to see that if S has an identity, then each  $T \in Mul_l(S)$  is inner.

Let **S** be a semigroup. Then **S** is called semisimple if  $I^2 = I$  for all ideal **I** of **S** (see [9], page 95 for more details).

**Theorem 2.4.** Let S be a semigroup whit an identity and  $T \in Mul_l(S)$ . If  $S_T$  is semisimple, then S is so. The converse is true if  $S_T$  is left cancellative and T is surjective.

**Proof.** Since S is unital there exists  $\mu \in S$  such that  $T = L_{\mu}$ . Suppose that  $S_T$  is semisimple and I is an ideal of S. Then

$$I \circ S = IT(S) \subseteq IS \subseteq I$$
.

Similarly,  $S \circ I \subseteq I$ . It follows that I is an ideal of  $S_T$ . By the hypothesis  $(I_T)^2 = I \circ I = I$ . Now, take  $r \in I$  then there are  $s, t \in I$  such that

$$r = s \circ t = sT(t) = s(\mu t) \in I^2$$
.

So we show that  $I^2 = I$  and hence S is semisimple.

Conversely, assume that  $S_T$  is left cancellative and  $T \in Mul_l(S)$  is surjective then by theorem 2.1(ii),  $T^{-1} \in Mul_l(S)$ . So, there exists  $b \in S$  such that  $T^{-1} = L_b$ . Suppose that  $\mathbf{\breve{S}} = S_{T^{-1}}$ . Then we have .

$$S = S_{TT^{-1}} = (S_T)_{T^{-1}} = \widecheck{S}_{T^{-1}}.$$

By hypothesis and above the proof,  $\mathbf{\breve{S}} = \mathbf{S}_{T^{-1}}$  is semisimple.

Semigroup Algebra 
$$\ell^1(S_T)$$

We say that a discrete semigroup S is amenable if there exists a positive linear functional on  $\ell^{\infty}(S)$  called a mean such that m(1) = 1 and  $m(l_s f) = m(f)$ ,  $m(r_s f) = m(f)$  for each  $s \in S$ , where  $l_s f(t) = f(st)$  and  $r_s f(t) = f(ts)$  for all  $t \in S$ . The definition of amenable group is similar to semigroup case. Refer to [12] for more details.

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Let  $\mathfrak A$  be a Banach algebra and let X be a Banach  $\mathfrak A$  —bimodule. A derivation from  $\mathfrak A$  to X is a linear map  $D: \mathfrak A \longrightarrow X$  such that

$$D(ab) = D(a) \cdot b + a \cdot D(b) \quad (a, b \in \mathfrak{A}).$$

A derivation D is inner if there exists  $x \in X$  such that

$$D(a) = a \cdot x - x \cdot a \qquad (a \in \mathfrak{A}).$$

The Banch algebra  $\mathfrak A$  is amenable if every bounded derivation  $D: \mathfrak A \to X^*$  is inner for all Banach  $\mathfrak A$  —bimodule X. Where  $X^*$  is the dual space of X. We say that the Banch algebra  $\mathfrak A$  is weakly amenable if any bounded derivation D from  $\mathfrak A$  to  $\mathfrak A^*$  is inner. Fore more details see [12], [16].

If **S** is a commutative semigroup, by theorem 5.8 of [8]  $\ell^1(S)$  is called semisimple if and only if for all  $x, y \in S$ ,  $x^2 = y^2 = xy$  implies x = y.

**Theorem 3.1.** Let S be a commutative semigroup and let  $T \in Mul_l(S)$  be injective. Then  $\ell^1(S)$  is semisimple if and only if  $\ell^1(S_T)$  is semisimple.

**Proof.** Take  $r, s \in S$ . Then  $r^2 = s^2 = rs$  if and only if  $T(r^2) = T(s^2) = T(r)T(s)$  or equivalently  $r_0 r = s_0 s = r_0 s$ , because T is injective. So, by theorem 5.8 [8],  $\ell^1(S)$  is semisimple if and only if  $\ell^1(S_T)$  is semisimple.

**Theorem 3.2.** Let S be a discrete semigroup and  $T \in Mul_l(S)$ . Then

- (i) The left multiplier T has an extension  $\tilde{T} \in Mul_l(\ell^1(S))$  with the norm decreasing.
- (ii) The left multiplier T is injective if and only if so is  $\tilde{T}$ .
- (iii) If T is injective then  $\tilde{T}$  is an isometry and also  $\ell^1(S_T)$  and  $(\ell^1(S))_T$  are isomorphic.

**Proof.** (i) An arbitrary element  $f \in \ell^1(S)$  is of the form  $f: S \to \mathbb{C}$  such that f(x) = 0 except at the most countable subset A of S. If A is a finite subset of S then  $f = \sum_{k=1}^n f(x_k) \, \delta_{x_k}$  for some fixed  $n \in \mathbb{N}$ . So in general we have

$$f = \sum_{x \in S} f(x)\delta_x = \sum_{x \in A} f(x)\delta_x = \sum_{k=1}^{\infty} f(x_k) \delta_{x_k}.$$

Now, for each  $n \in \mathbb{N}$ , let  $f_n = \sum_{k=1}^n f(x_k) \delta_{x_k}$  and define  $\tilde{T}: \ell^1(S) \to \ell^1(S)$  by

$$\widetilde{T}(\delta_x) = \delta_{T(x)} \qquad (x \in \mathbf{S}), 
\widetilde{T}(f_n) = \sum_{k=1}^n f(x_k) \widetilde{T}(\delta_{x_k}) = \widecheck{f_n}.$$

For each  $m, n \in \mathbb{N}$  where  $n \ge m$ , we have

$$\begin{aligned} \left\| \widetilde{T}(f_n) - \widetilde{T}(f_m) \right\|_1 &= \left\| \widetilde{f_n} - \widetilde{f_m} \right\|_1 = \left\| \sum_{k=m}^{k=n} f(x_k) \ \widetilde{T}(\delta_{x_k}) \right\| = \left\| \sum_{k=m}^{k=n} f(x_k) \ \delta_{T(x_k)} \right\| \\ &\leq \sum_{k=m}^{k=n} |f(x_k)| = \|f_n - f_m\|_1 \,. \end{aligned}$$

So  $\{\tilde{T}(f_n)\}_n$  is a Cauchy sequence and it is convergent. Now, we define  $\tilde{T}(f) = \lim_n \tilde{f_n}$ . Then the definition is well defined. Hence

$$\widetilde{T}(f) = \sum_{k=1}^{\infty} f(x_k) \widetilde{T}(\delta_{x_k}) = \widetilde{f}$$
,

also

$$\|\tilde{f}\|_{1} \le \sum_{x_{k} \in A} |f(x_{k})| = \|f\|_{1} \text{ or } \|\tilde{T}(f)\|_{1} \le \|f\|_{1}.$$

It shows that  $\tilde{T}$  is norm decreasing.

In the following, we extend  $\tilde{T}$  by linearity. Let  $f, g \in \ell^1(S)$ . Then there are two at most countable sub set A, B of S such that

$$f = \sum_{x \in A} f(x) \delta_x$$
,  $g = \sum_{x \in B} g(x) \delta_x$ .

Suppose that  $D = A \cup B$ . So we have  $f + g = \sum_{x \in D} (f(x) + g(x))\delta_x$ .

Then, it follows that

$$\widetilde{T}(f+g) = \widetilde{f+g} = \sum_{x \in D} (f(x) + g(x)) \widetilde{T}(\delta_x) = \sum_{x \in A} f(x) \, \widetilde{T}(\delta_x) + \sum_{x \in B} g(x) \, \widetilde{T}(\delta_x)$$

$$= \widetilde{f} + \widetilde{g}.$$

Also, if  $\alpha \in \mathbb{C}$ , we have

$$\widetilde{T}(\alpha f) = \widetilde{\alpha f} = \sum_{x \in A} \alpha f(x) \widetilde{T}(\delta_x) = \alpha \sum_{x \in A} f(s) \widetilde{T}(\delta_x) = \alpha \widetilde{T}(f).$$

Therefore,  $\tilde{T}$  is a bounded linear isometry.

Now, we prove that  $\tilde{T} \in Mul_l(\ell^1(S))$ . Take  $x, y \in S$ . Then

$$\tilde{T}\big(\delta_x*\delta_y\big)=\tilde{T}\big(\delta_{xy}\big)=\delta_{T(xy)}=\delta_{T(x)y}=\delta_{T(x)}*\delta_y=\tilde{T}(\delta_x)*\delta_y.$$

Let  $y \in S$  be fixed and f,  $g \in \ell^1(S)$ . Then

$$\begin{split} \tilde{T}\big(f * \delta_y\big) &= \tilde{T}\big(\sum_{x \in A} f(x) \, \delta_{xy}\,\big) = \sum_{x \in A} f(x) \tilde{T}\big(\delta_{xy}\big) \\ &= \Big(\sum_{x \in A} \tilde{T} \, (\delta_x)\Big) * \delta_y = \tilde{f} * \delta_y = \tilde{T}(f) * \delta_y. \end{split}$$

In the general case, we have

$$\begin{split} \tilde{T}(f*g) &= \tilde{T}(\sum_{x \in A} f(x) \left(\sum_{y \in B} g(y)\right) \delta_{xy}) = \sum_{x \in A} f(x) \sum_{y \in B} g(y) \tilde{T}(\delta_x) * \delta_y \\ &= \sum_{x \in A} f(x) \tilde{T}(\delta_x) * \sum_{y \in B} g(y) \delta_y = \tilde{T}(f) * g \end{split}.$$

This shows that  $\tilde{T}$  is a multiplier on  $\ell^1(S)$ .

(ii) Let T be injective. Take  $x, y \in S$  and suppose that  $\tilde{T}(\delta_x) = \tilde{T}(\delta_y)$ . Then  $\delta_{T(x)} = \tilde{T}(\delta_x) = \tilde{T}(\delta_y) = \delta_{T(y)}$ .

Therefore, T(x) = T(y). Since T is injective, we have x = y. It follows that  $\delta_x = \delta_y$ , consequently  $\tilde{T}$  is injective.

Conversely, the same argument shows that the converse holds.

(iii) Let T be injective and  $f \in \ell^1(S)$ . Then there exists at most a countable subset  $A \subseteq S$  such that

$$f = \sum_{x \in A} f(x) \delta_x$$

Since A and T(A) have the same cardinal number,  $\|\tilde{T}(f)\| = \|\sum_{x \in A} f(x) \delta_x\| = \sum_{x \in A} |f(x)| = \|f\|_1$ , so  $\tilde{T}$  is an isometry.

Now, we can define a new multiplication "\*" on  $\ell^1(S)$  as follow

$$f * g = f * \tilde{T}g \quad (f, g \in \ell^1(S)).$$

By a similar argument in theorem1.31 [10],  $\ell^1(S)$  with the new product is a Banach algebra that is denoted it by  $\ell^1(S)_{\tilde{T}}$ . We define the map  $\Psi: \ell^1(S_T) \to \ell^1(S)_{\tilde{T}}$ , by

$$\Psi(\delta_x) = \delta_x \qquad (x \in S).$$

Take x,  $y \in S$ . Then

$$\Psi(\delta_{x} * \delta_{y}) = \Psi(\delta_{x \circ y}) = \delta_{xT(y)} = \delta_{x} * \delta_{T(y)}$$
$$= \delta_{x} * \tilde{T}(\delta_{y}) = \delta_{x} * \delta_{y}$$
$$= \Psi(\delta_{x}) * \Psi(\delta_{y}).$$

So, in general case, we have

$$\Psi(f * g) = \Psi(f) * \Psi(g) \qquad (f, g \in \ell^1(S)).$$

Thus,  $\Psi$  is an isomorphism. Therefore  $\ell^1(S_T)$  and  $\ell^1(S)_{\tilde{T}}$  are isomorphic

**Theorem 3.3**. Let **S** be a semigroup and  $T \in Mul_l(S)$  be bijective. Then  $\ell^1(S)$  is amenable if and only if  $\ell^1(S_T)$  is amenable.

**Proof.** By theorem 3.2, we have  $\ell^1(S_T) \cong \ell^1(S)_{\tilde{T}}$ . Suppose that  $\ell^1(S_T)$  is amenable and define  $\varphi: \ell^1(S)_{\tilde{T}} \to \ell^1(S)$  by  $\varphi(f) = \tilde{T}(f)$ . Take  $x, y \in S$ . Then

$$\varphi(\delta_x * \delta_y) = \tilde{T}(\delta_x * \delta_y) = \tilde{T}(\delta_{xT(y)}) = \tilde{T}(\delta_x * \delta_{T(y)}) = \tilde{T}(\delta_x) * \delta_{T(y)}$$
$$= \tilde{T}(\delta_x) * \tilde{T}(\delta_y) = \varphi(\delta_x) * \varphi(\delta_y).$$

Now, by induction and continuity of  $\tilde{T}$ , we have

$$\varphi(f | \overline{*} | g) = \varphi(f) * \varphi(g)$$
.

If T is bijective,  $\tilde{T}$  is bijective. Therefore  $\varphi$  is an epimorphism of  $\ell^1(S_T)$  onto  $\ell^1(S)$ .

Hence, by proposition 2.3.1 [16]  $\ell^1(S)$  is amenable.

Conversely, suppose that  $\ell^1(S)$  is amenable. Since T is bijective,  $\tilde{T}$  is bijective. Therefore  $\tilde{T}^{-1}$  exists. Now define  $\theta: \ell^1(S) \to \ell^1(S_T)$  [ $\cong \ell^1(S)_{\tilde{T}}$ ] by  $\theta(f) = \tilde{T}^{-1}(f)$ .

Take  $x, y \in S$ . Then

$$\theta(\delta_x * \delta_y) = \tilde{T}^{-1}(\delta_{xy}) = \tilde{T}^{-1}(\delta_x)\tilde{T}\tilde{T}^{-1}(\delta_y) = \tilde{T}^{-1}(\delta_x) * \tilde{T}^{-1}(\delta_y)$$
$$= \theta(\delta_x) * \theta(\delta_y).$$

Similarly  $\theta$  is an epimorphism from  $\ell^1(S)$  onto  $\ell^1(S_T)$ . By proposition 2.3.1 [16]  $\ell^1(S_T)$  is amenable.

Note that, in general, it is not known when  $\ell^1(S)$  is weakly amenable. For more detials see [2].

**Theorem3.4.** Let **S** be a semigroup and  $T \in Mul_l(S)$  be bijective. Then, if **S** is completely regular then  $\ell^1(S_T)$  is weakly amenable.

**Proof.** It is enough to prove that  $S_T$  is completely regular, then by theorem 3.6 [2],  $\ell^1(S_T)$  can be weakly amenable. Take  $s \in S$ . Then there exists  $r \in S$  such that T(s) = T(s)T(r)T(s), T(r)T(s) = T(s)T(r), since T is bijective and S = T(S) is completely regular. So we have  $T(s) = T(s \circ r \circ s)$  and  $T(r \circ s) = T(s \circ r)$ . Hence  $s = s \circ r \circ s$  and  $r \circ s = s \circ r$  for some  $r \in S$ , since T is injective. Therefore  $S_T$  is completely regular.

**Corollary.3.5.** Suppose that **S** is a commutative completely regular semigroup and  $T \in Mul_l(S)$  is injective. Then  $\ell^1(T(S)_T)$  is weakly amenable.

**Proof.** [2, theorem 3.6]  $\ell^1(S)$  is weakly amenable. Define  $\varphi: S \to \ell^1(S)_T$  by

$$\varphi(s) = T^{-1}(s)$$
  $(s \in S)$ .

We show that  $\varphi$  is a homomorphism . Take  $s \in S$ , then we have

$$\varphi(rs) = T^{-1}(rs) = T^{-1}(r)s = T^{-1}(r) \circ (T^{-1}s)$$
.

So  $\varphi$  is a homomorphism. Then by proposition 2.1[7],  $\ell^1(T(S)_T)$  is weakly amenable. In the case that S is a group, it is easy to see that the amenability of S implies the amenability of  $\ell^1(S_T)$ . Indeed, when S is a group, by theorem 2.1,  $S_T$  is a semigroup and one can easily prove that  $S_T$  is also a group. On the other hand,  $Mul_l(S) \cong S$  because S is a unital semigroup, so each  $T \in Mul_l(S)$  is inner and of the form  $T = L_S$  for some  $S \in S$ . Also  $T^{-1} = L_{a^{-1}}$  exists, since S is a group. Then the map  $\theta: S_T \to S$  defined by  $\theta(S) = T(S)$  is an isomorphism; that is  $S \cong S_T$ . Thus we have the following result:

**Corollary 3.6.** Let S be a cancellative regular discrete semigroup. Then  $\ell^1(S)$  is amenable if and only if  $\ell^1(S_T)$  is amenable.

**Proof.** By [9, Exercise 2.6.11] S is a group. So the assertion holds by [15, theorem 2.1.8]

## **Examples**

In this section we present some examples which either comments on our results or indicate necessary condition in our theorems.

**4.1.** There are semigroups S and  $T \in Mul_l(S)$  such that the background semigroups S are not commutative but their induced semigroups  $S_T$  are commutative.

This example shows that the condition  $\overline{T(S)} = S$ , in theorem 2.2, can not be omitted.

Let S be the set  $\{a, b, c, d, e\}$  with operation table given by

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	a	b	c	d	e
a	a	a	a	d	d
b	a	b	c	d	d
c	a	c	b	d	d
d	d	d	d	a	a
e	d	e	e	a	a

Clearly(S,.) is a non-commutative semigroup. Now, put  $T = L_a$  where  $L_a(x) = ax$  for all  $x \in S$ . One can get easily the operation table of  $S_T$  as fallow:

0	a	b	c	d	e
a	a	a	a	d	d
b	a	a	a	d	d
с	a	a	a	d	d
d	d	d	d	a	a
e	d	d	d	a	a

The operation table shows that the induced semigroup  $S_T$  is commutative and  $T(S) \neq S$ . Also the other induced semigroup  $S_T$  is commutative for  $T = L_d$  analogously.

Now we present some important theorems from [14] that we need in the following examples:

**Theorem 4. 2.** Let S be a semigroup. Suppose that  $\ell^1(S)$  is amenable. Then

- (i)  $\boldsymbol{S}$  is amenable
- (ii) S is regular.
- (iii) E(S) is finite.
- (iv)  $\ell^1(S)$  has an identity.

**Proof.** (i) That is lemma 3 in [5].

(ii) and (iii) See theorem 2 in [6].

(iv) That is corollary 10.6 in [4].

**Theorem 4.3.** Let S be a finite semigroup. Then the following statements are equivalent:

- (i)  $\ell^1(S)$  is amenable.
- (ii) S is regular and  $\ell^1(S)$  is nuital.
- (ii) ) S is regular and  $\ell^1(S)$  is semisimple.

**Proof.** Refer to [3].

**4.4.** There are semigroups S and  $T \in Mul_l(S)$  such that S and  $\ell^1(S)$  are amenable but  $S_T$  is not regular and also,  $\ell^1(S_T)$  is not amenable.

This example shows that two semigroup algebras  $\ell^1(S)$  and  $\ell^1(S_T)$  can be different in some properties. Also, it notifies that the bijectivity of T in the theorem 3.3 is essential. Put  $S = \{x_0, x_1, x_2, ..., x_n\}$  with the operation  $x_i x_j = x_{Max\{i,j\}}$   $(0 \le i, j \le n, n \ge 2)$ .

Then S is a semigroup. Since

$$Max\{i, Max\{j, k\}\} = Max\{Max\{i, j\}, k\} = Max\{i, j, k\}.$$

We denote it by  $S_V$ . This semigroup is commutative. So by (0.18) in [12], it is amenable.  $S_V$  is a unital semigroup and has a zero; indeed,  $e_s = x_0$  and  $o_s = x_n$ . Also, it is a regular semigroup and  $Mul(S_V) \cong S_V$  because  $S_V$  has an identity.

Evidently,  $S_V$  is regular since each  $s \in S_V$  is idempotent. The semigroup algebra  $\ell^1(S_V)$  is a unital algebra because  $S_V$  has an identity. So by theorem 4.3 (ii)  $\ell^1(S_V)$  is amenable.

Now, take  $T = L_{x_k}$  for a fixed  $x_k \in S$  where  $k \ge 1$ . By theorem 2.2,  $(S_v)_T$  is commutative so is amenable. We show that T is neither injective and nor surjective.

Take  $x_i \in S_{\vee}$ , then  $Tx_i = x_k x_i = x_{max\{k,i\}}$ . So

$$T(\mathbf{S}_{\vee}) = \{x_k, x_{k+1}, \dots, x_n\} \neq \mathbf{S}_{\vee}.$$

Hence, T is not surjective.

Again, take distinct elements  $x_i, x_j$  in  $S_V$  for some i, j < k such that  $T(x_i) = T(x_j)$ . Then we have  $x_{max\{k,i\}} = x_{max\{k,j\}}$  but  $x_i \neq x_j$ . So T is not injective.

We prove that  $(S_{\vee})_T$  is not regular. If  $(S_{\vee})_T$  is regular, then for  $x_{k-1} \in S_{\vee}$  there exists an element  $x_j \in S_{\vee}$  such that

$$x_{k-1} = x_{k-1} \circ x_{i} \circ x_{k-1} = x_{Max\{k,i\}}$$
.

That implies that  $max\{k,j\} = k-1$ ; which is impossible. Consequently, by theorem 4.2 (ii) or 4.3 (ii),  $\ell^1((S_V)_T)$  is not amenable.

Also, the inequality  $S_{V} \circ S_{V} = \{x_{k}, x_{k+1}, ..., x_{n}\} \neq S_{V}$  shows that  $\ell^{1}((S_{V})_{T})$  is not weakly amenable. In the next example we show that in the theorem 3.2 (iii) the condition "injectivity of " can not be omitted.

**4.5** There are a semigroup S and  $T \in Mul_l(S)$  such that  $T \in Mul_l(S)$  is not injective and the corresponding  $\tilde{T} \in Mul_l(l^1(S_T))$  is not an isometry.

Suppose that  $S_V$  is a semigroup as in example 4.4 and  $T = L_{x_k}$  for some fixed 1 < k < n. If  $f \in \ell^1(S_V)$  then  $f = \sum_{i=0}^n f(x_i) \delta_{x_i}$  and also  $\tilde{T}(f) = \sum_{i=0}^n f(x_i) \delta_{T(x_i)}$ . But  $T(x_i) = \begin{cases} x_i & k < i \leq n \\ x_k & 0 \leq i \leq k \end{cases}$ ,

SO

$$\tilde{T}(f) = \left(\sum_{i=0}^k f(x_i)\right) \delta_{x_k} + \sum_{i=k+1}^n f(x_i) \delta_{T(x_i)}.$$

Hence

$$\|\tilde{T}(f)\| = \left| \sum_{i=0}^{k} f(x_i) \right| + \sum_{i=k+1}^{n} |f(x_i)|$$

$$\leq \sum_{i=0}^{k} |f(x_i)| + \sum_{i=k+1}^{n} |f(x_i)| = \|f\|_1,$$

It shows that  $\tilde{T}$  is not an isometry.

**4.6.** There are semigroups S and  $T \in Mul_l(S)$  such that  $\ell^1(S)$  is semisimple. But  $\ell^1(S_T)$  is not semisimple. This example remind that, in theorem 3.1 the multiplier T must be injective.

Let **S** be a set  $\{x_0, x_1, ..., x_n\}$  where  $n \in \mathbb{N}$  and  $n \ge 3$  is fixed. by operation given by  $xy = x_{min\{i,j\}}$ , **S** is a commutative semigroup. Since

$$min\{i, min\{j, k\}\} = min\{min\{i, j\}, k\} = min\{i, j, k\} \qquad (i, j, k \in \mathbf{N})$$

We denote it briefly by  $S_{\wedge}$  For each  $x, y \in S$  the equality  $x^2 = y^2 = xy$  implies x = y. So by Theorem 5.8 [8]  $\ell^1(S_{\wedge})$  is semisimple.

Now, let  $T = L_{x_k}$  for a fixed  $1 \le k < n - 1$ . It is easy to see that  $T(x_k) = T(x_n)$  but  $x_k \ne x_n$ . So the multiplier T is not injective.

We show that neither  $\mathbf{S}_{\wedge}$  nor  $\ell^1(\mathbf{S}_{\wedge})_T$  is semisimple.

Each ideal of **S** is of the form

$$I_m = \{x_0, x_1, \dots, x_m\} \quad (m \le n).$$

We claim that  $S_T$  is not semisimple. Since for each  $m \in \mathbb{N}$  we have

$$I_m \circ I_m = \begin{cases} I_m & m \le k \\ I_k & m > k \end{cases}.$$

On the other hand, for each  $x_i, x_j \in S$  where  $i \neq j$  and i, j > k, we have  $x_i \circ x_i = x_j \circ x_j = x_i \circ x_j = x_k$ , while  $x_i \neq x_j$ . Thus, Theorem 5.8 [8] shows that  $\ell^1(S_{\wedge})_T$  is not semisimple.

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