Amenability and Weak Amenability of the Semigroup Algebra $\ell^1(S_T)$

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Abstract

Let **S** be a semigroup with a left multiplier T on **S**. A new product on **S** is defined by T related to **S** and T such that **S** and the new semigroup $\mathbf{S_T}$ have the same underlying set as **S**. It is shown that if T is injective then $\ell^1(\mathbf{S_T}) \cong \ell^1(\mathbf{S})_{\widetilde{T}}$ where, $\widetilde{T}$ is the extension of T on $\ell^1(\mathbf{S})$. Also, we show that if T is bijective, then $\ell^1(\mathbf{S})$ is amenable if and only if $\ell^1(\mathbf{S_T})$ is so. Moreover, if **S** completely regular, then $\ell^1(\mathbf{S_T})$ is weakly amenable.

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Introduction

Let S be a semigroup and T be a left multiplier on S. We present a general method of defining a new product on S which makes S a semigroup. Let S_T denote S with the new product. These two semigroups are sometims different and we try to find conditions on S and T such that the semigroups S and S_T have the same properties. This idea has started by Birtel in [1] for Banach algebras and continued by Larsen in [11]. Recently, this notion developed by some authors, for more details see [1], [10], [11], [12] and [15]. One of the best result in this work expresses that $L^1(G)_T$ is Arens regular if and only if G is a compact group [10]. We continue this direction on the regularity of S and S_T and the amenability of their semigroup algebras.

The term of semigroup will be a non-empty set S endowed with an associative binary operation on S , defined by $(s, t) \to st$. If S is also a Hausdorff topological space and the binary operation is jointly continuous, then S is called a topological semigroup.

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Let $p \in S$. Then p is an idempotent if $p^2 = p$. The set of all idempotents of S is denoted by $E(S)$.

An element e is a left (right) identity if $es = s$ (resp. $se = s$) for all $s \in S$. An element $e\epsilon S$ is an identity if it is a left and a right identity. An element z is a left (resp. right) zero if $zs = z$ (resp. $sz = z$) for all $s \in S$. An element $z\epsilon S$ is a zero if it is a left and a right zero. We denote any zero of S by 0_S (or z_S). An element $p\epsilon S$ is a regular element of S if there exists $t\epsilon S$ such that $p = ptp$ and p is completely regular if it is regular and $pt = tp$. We say that $p \in S$ has an inverse if there exists $t\epsilon S$ such that $p = ptp$ and $t = tpt$. Note that the inverse of element $p \in S$ need not be unique. If $p\epsilon S$ has an inverse, then p is regular and vise versa. Since, if $p \in S$ is regular, there exists $s \in S$ such that $p = psp$. Let $t = sps$. Then

$$p = psp = (psp)sp = p(sps)p = ptp\ , \ t = sps = s(psp)s = (sps)p(sps) = tpt\,.$$

So p has an inverse. We say that S is a regular (resp. completely regular) semigroup if each $p\epsilon S$ is regular (resp. completely regular). Also S is an inverse semigroup if each $p \in S$ has a unique inverse. The map $T : S \to S$ is called a left (resp. right) multiplier if

$$T(st) = T(s)t \quad (\text{resp. } T(st) = sT(t)) \qquad (s, t\epsilon S).$$

The map $T : S \to S$ is a multiplier if it is a left and right multiplier.Let S be a topological semigroup. The net $(e_\alpha) \subseteq S$ is a left (resp. right) approximate identity if $\lim_\alpha e_\alpha t = t$. (resp. $\lim_\alpha t\, e_\alpha = t$) $(t\epsilon S)$. The net $(e_\alpha) \subseteq S$ is an approximate identity if it is a left and a right approximate identity.

Let S be a discrete semigroup. We denote by $\ell^1(S)$ the Banach space of all complex function $f: S \to \mathbb{C}$ having the form

$$f = \textstyle\sum_{s\epsilon S} f(s)\delta_s\ ,$$

such that $\sum_{s\epsilon S}|f(s)| = \|f\|_1$ is finite, where δ_s is the point mass at $\{s\}$.For $f, g\epsilon\ell^1(S)$ we define the convolution product on $\ell^1(S)$ as fallow:

$$f * g(s) = \textstyle\sum_{t_1 t_2 = s} f(t_1)g(t_2) \qquad (s\epsilon S),$$

with this product $\ell^1(S)$ becomes a Banach algebra and is called the semigroup algebra on S.

Remark 1.1. If $f\epsilon\ell^1(S)$ then $f = 0$ on S except at most on a countable subset of S. In other words, the set $A = \{s\epsilon S :\ f(s) \neq 0\}$ is at most countable. Since, if $A_n = \left\{s\epsilon S :\ |f(s)| \geq \frac{1}{n}\right\}$, $A = \bigcup_{n\epsilon N} A_n$. Set $\|f\|_1 = M$ and $n\epsilon N$ is fixed. Then we have

$$M = \sum_{s\epsilon S}|f(s)| \geq \sum_{s\epsilon A_n}|f(s)| \geq \sum_{s\epsilon A_n}\frac{1}{n} = \frac{1}{n}|A_n|\ ,$$

where $|A_n|$ is the cardinality of A_n . So $|A_n| \leq nM$. Hence A_n is a finite subset of S and thus A is at most countable.

Semigroup S_T

Let $T \in Mul_l(\boldsymbol{S})$. Then we define a new binary operation "$\circ$" on $\boldsymbol{S}$ as follow :

$$s \circ t = s\,T(t) \quad (s, t \epsilon \boldsymbol{S}).$$

The set S equipt with the new operation "$\circ$" is denoted by S_T and sometimes called "induced semigroup of S" . Now we have the following results.

Theorem 2.1. Let **S** be a Semigroup. Then (i) if $T \in Mul_l(\boldsymbol{S})$ then $\boldsymbol{S}_T$ is a semigroup . The converse is true if **S** is left cancellative and T is surjective .

(ii) If $\boldsymbol{S}_T$ is left cancellative and T is surjective, then $T^{-1} \in Mul_l(\boldsymbol{S})$.

(iii) If **S** is a topological semigroup and $\boldsymbol{S}_T$ has a left approximate identity then $T^{-1} \in Mul_l(\boldsymbol{S})$.

Proof. i) Let $T \epsilon Mul_l(\boldsymbol{S})$ and take r,s,t ϵ**S**. Then

$$r \circ (s \circ t) = r\,T(s \circ t) = r\,T(s\,T(t)) = r\,T(s)T(t) = (r\,T(s))\,T(t)$$
$$= (r \circ s) \circ t$$

So, $\boldsymbol{S}_T$ is a semigroup.

Conversely, suppose that **S** is left cancellative and take r,s,t ϵ**S**. Since T is surjective, there exists $u \epsilon \boldsymbol{S}$ such that $T(u) = t$. Then

$$rT(st) = rT(sT(u)) = r \circ (s \circ u) = (r \circ s) \circ u = (rT(s))T(u)$$
$$= r(T(s)t).$$

By the left cancellativity of **S**, we have $T(st) = T(s)t \quad (r, s \epsilon \boldsymbol{S})$. So, T is a left multiplier.

ii) We must prove that T is injective. To do this end, take r,s,uϵ**S** and let $T(r) = T(s)$. Then $u \circ r = uT(r) = uT(s) = u \circ s$. So $r = s$, since $\boldsymbol{S}_T$ is left cancellative. Hence T^{-1} exists.

Now, we show that $T^{-1} \in Mul_l(\boldsymbol{S})$. Take $r, s \epsilon \boldsymbol{S}$. Then

$$T^{-1}(rs) = T^{-1}[TT^{-1}(r)s] = T^{-1}[T(T^{-1}(r)s)]$$
$$= (T^{-1}T)[T^{-1}(r)s] = T^{-1}(r)s\,.$$

iii) It is enough to show that T is injective. Take $r, s \epsilon \boldsymbol{S}$ and suppose that $T(r) = T(s)$. Then

$$r = \lim_\alpha e_\alpha \circ r = \lim_\alpha e_\alpha T(r) = \lim_\alpha e_\alpha T(s) = \lim_\alpha e_\alpha \circ s = s.$$

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There are many properties that induced from **S to** semigroup $\mathbf{S}_\mathrm{T}$. But sometimes they are different.

Theorem2.2. Let **S** be a Hausdorff topological semigroup and $\mathrm{T} \epsilon \mathrm{Mul}_l(\mathbf{S})$. If **S** is commutative then so is $\mathbf{S}_\mathrm{T}$. The converse is true if $\overline{\mathrm{T}(\mathbf{S})} = \mathbf{S}$.

Proof. Suppose **S** is commutative and take $\mathrm{r}, \mathrm{s} \in \mathbf{S}$. Then

$$\mathrm{r} \circ \mathrm{s} = \mathrm{r}\,\mathrm{T}(\mathrm{s}) = \mathrm{T}(\mathrm{s})\mathrm{r} = \mathrm{T}(\mathrm{sr}) = \mathrm{T}(\mathrm{rs}) = \mathrm{T}(\mathrm{r})\,\mathrm{s} = \mathrm{s}\mathrm{T}(\mathrm{r}) = \mathrm{s} \circ \mathrm{r}\,.$$

So, $\mathbf{S}_\mathrm{T}$ is commutative.

Conversely, Let $\mathbf{S}_\mathbf{T}$ be commutative and take $\mathrm{r}, \mathrm{s}\, \epsilon \mathbf{S}$. Then there exist nets (r_α) and (s_β) in **S** such that $\lim_\alpha \mathrm{T}(\mathrm{r}_\alpha) = \mathrm{r}$ and $\lim_\beta \mathrm{T}(\mathrm{s}_\beta) = \mathrm{s}$.

So, we have

$$\mathrm{rs} = \lim_\alpha \lim_\beta \mathbf{T}(\mathrm{r}_\alpha \circ \mathrm{s}_\beta) = \lim_\propto \lim_\beta \mathbf{T}(\mathrm{s}_\beta \circ \mathrm{r}_\alpha) = \lim_\alpha \lim_\beta \mathbf{T}(\mathrm{s}_\beta)\ \mathbf{T}(\mathrm{r}_\alpha) = \mathrm{s}\,\mathrm{r}.$$

Thus **S** is commutative .

In the sequel, we investigate some relations between two semigroup **S** and $\mathbf{S}_\mathbf{T}$ according to the role of the left multiplier T.

Theorem 2.3. Let **S** be a semigroup and $\mathrm{T} \epsilon \mathrm{Mul}_l(\mathbf{S})$ .Then

(i) If T is surjective and $\mathbf{S}_\mathbf{T}$ is an inverse semigroup then **S** is an inverse semigroup and $\mathrm{T}(\mathrm{s}^{-1}) = \mathrm{T}(\mathrm{s})^{-1}$ for all $\mathrm{s}\epsilon\mathbf{S}$.

(ii) If $\mathbf{S}_\mathrm{T}$ is an inverse semigroup and T is injective then $\mathrm{T}(\mathbf{S})$ is an inverse subsemigroup of **S**.

(iii) If T is bijective then $\mathbf{S}_\mathrm{T}$ is an inverse semigroup if and only if **S** is an inverse semigroup.

Proof. i) Suppose that $\mathbf{S}_\mathrm{T}$ is an inverse semigroup and T is surjective. Define the map $\varphi: \mathbf{S}_\mathrm{T} \to \mathbf{S}$ by $\varphi(\mathrm{s}) = \mathrm{T}(\mathrm{s})$. Take $\mathrm{r}, \mathrm{s}\epsilon\mathbf{S}$, then

$$\varphi(r \circ s) = T(r \circ s) = T(r)T(s) = \varphi(r)\varphi(s).$$

So, φ is an epimorphism from $\boldsymbol{S}_T$ onto **S**, since T is surjective. By theorem 5.1.4[**7**], **S** is an inverse semigroup and $T(s^{-1}) = T(s)^{-1}$ for all $s\epsilon\boldsymbol{S}$.

ii) Suppose that T is injective and $\boldsymbol{S}_T$ is an inverse semigroup. Evidently, $T(\boldsymbol{S})$ is a subsemigroup of $\boldsymbol{S}$. We show that it is an inverse semigroup. Take $s\epsilon T(\boldsymbol{S})$. There exists $t\epsilon\boldsymbol{S}$ such that $s = T(t)$. Also, there exists a unique element $u\epsilon\boldsymbol{S}$ such that $t = t\ {}_0\, u\ {}_0\, t$, since $\boldsymbol{S}_T$ is an inverse semigroup. Therefore, $T(t) = T(t)T(u)T(t)$, or $s = s\ {}_0\, T(u)\ {}_0\, s$. Of course, $T(u)$ is unique because $u\epsilon\boldsymbol{S}$ is unique and T is injective. Hence $T(\boldsymbol{S})$ is an inverse subsemigroup of $\boldsymbol{S}$.

iii) Suppose that T is bijective and let $\boldsymbol{S}_T$ be an inverse semigroup. Since T is injective and surjective, by (i) and (ii), $\boldsymbol{S} = T(\boldsymbol{S})$ is an inverse semigroup.
Conversely, suppose that $\boldsymbol{S}$ is an inverse semigroup. Since T is bijective , by theorem 2.1(ii), $T^{-1} \epsilon Mul_l(\boldsymbol{S})$. So φ^{-1}: $\boldsymbol{S} \longrightarrow \boldsymbol{S}_T$ defined by $\varphi^{-1}(s) = T^{-1}(s)$ is an epimorphism. Hence by (i) $\boldsymbol{S}_T$ is an inverse semigroup

We say that $T \in Mul_l(\boldsymbol{S})$ is an inner left multiplier if it has the form $T = L_s$ for some $s\epsilon\boldsymbol{S}$ where $L_s(t) = s\,t \qquad (t\epsilon\boldsymbol{S})$.

If $T\epsilon Mul_l(\boldsymbol{S})$ is inner, then each ideal of $\boldsymbol{S}$ is permanent under T; that is $T(\boldsymbol{I}) \subseteq \boldsymbol{I}$ for all ideal **I** of $\boldsymbol{S}$. It is easily to see that if $\boldsymbol{S}$ has an identity, then each $T \in Mul_l(\boldsymbol{S})$ is inner.

Let **S** be a semigroup. Then **S** is called semisimple if $\boldsymbol{I}^2 = \boldsymbol{I}$ for all ideal **I** of **S** (see [**9**], page 95 for more details).

Theorem 2.4. Let $\boldsymbol{S}$ be a semigroup whit an identity and $T \in Mul_l(\boldsymbol{S})$. If $\boldsymbol{S}_T$ is semisimple, then $\boldsymbol{S}$ is so**.** The converse is true if $\boldsymbol{S}_T$ is left cancellative and T is surjective.

Proof. Since $\boldsymbol{S}$ is unital there exists $\mu\epsilon\boldsymbol{S}$ such that $\mathrm{T} = L_\mu$. Suppose that $\boldsymbol{S}_T$ is semisimple and $\boldsymbol{I}$ is an ideal of $\boldsymbol{S}$. Then

$$\boldsymbol{I} \circ \boldsymbol{S} = \boldsymbol{I}\, T(\boldsymbol{S}) \subseteq \boldsymbol{I}\,\boldsymbol{S} \subseteq \boldsymbol{I}\ .$$

Similarly, $\boldsymbol{S} \circ \boldsymbol{I} \subseteq \boldsymbol{I}$. It follows that $\boldsymbol{I}$ is an ideal of $\boldsymbol{S}_T$. By the hypothesis $(\boldsymbol{I}_T)^2 = \boldsymbol{I} \circ \boldsymbol{I} = \boldsymbol{I}$**.** Now, take $r\epsilon\boldsymbol{I}$ then there are $s, t\epsilon\boldsymbol{I}$ **s**uch that

$$r = s \circ t = sT(t) = s(\mu t) \in \boldsymbol{I}^2.$$

So we show that $\boldsymbol{I}^2 = \boldsymbol{I}$ and hence $\boldsymbol{S}$ is semisimple.

Conversely, assume that $\boldsymbol{S}_T$ is left cancellative and $T \in Mul_l(\boldsymbol{S})$ is surjective then by theorem 2.1(ii), $T^{-1}\epsilon Mul_l(\boldsymbol{S})$. So, there exists $b\epsilon\boldsymbol{S}$ such that $T^{-1} = L_b$. Suppose that $\breve{\boldsymbol{S}} = \boldsymbol{S}_{T^{-1}}$. Then we have .

$$\boldsymbol{S} = \boldsymbol{S}_{TT^{-1}} = (\boldsymbol{S}_{\boldsymbol{T}})_{T^{-1}} = \breve{\boldsymbol{S}}_{T^{-1}}.$$

By hypothesis and above the proof, $\breve{\boldsymbol{S}} = \boldsymbol{S}_{T^{-1}}$ is semisimple.

Semigroup Algebra $\ell^1(\boldsymbol{S}_{\boldsymbol{T}})$

We say that a discrete semigroup $\boldsymbol{S}$ is amenable if there exists a positive linear functional on $\ell^\infty(\boldsymbol{S})$ called a mean such that $m(\mathbf{1}) = 1$ and $m(l_s f) = m(f)$, $m(r_s f) = m(f)$ for each $s\epsilon\boldsymbol{S}$, where $l_s f(t) = f(st)$ and $r_s f(t) = f(ts)$ for all $t\epsilon\boldsymbol{S}$. The definition of amenable group is similar to semigroup case. Refer to [**12**] for more details.

Let $\mathfrak{A}$ be a Banach algebra and let X be a Banach $\mathfrak{A}$ –bimodule. A derivation from $\mathfrak{A}$ to X is a linear map $D: \mathfrak{A} \longrightarrow X$ such that

$$D(ab) = D(a) \cdot b + a \cdot D(b) \qquad (a, b \in \mathfrak{A}).$$

A derivation D is inner if there exists $x \epsilon X$ such that

$$D(a) = a \cdot x - x \cdot a \qquad (a \in \mathfrak{A}).$$

The Banch algebra $\mathfrak{A}$ is amenable if every bounded derivation $D: \mathfrak{A} \longrightarrow X^*$ is inner for all Banach $\mathfrak{A}$ –bimodule X. Where X^* is the dual space of X. We say that the Banch algebra $\mathfrak{A}$ is weakly amenable if any bounded derivation D from $\mathfrak{A}$ to $\mathfrak{A}^*$ is inner. Fore more details see [12] , [16] .

If $\boldsymbol{S}$ is a commutative semigroup, by theorem 5.8 of [**8**] $\ell^1(\boldsymbol{S})$ is called semisimple if and only if for all $x, y \epsilon \boldsymbol{S}$, $x^2 = y^2 = xy$ implies $x = y$.

Theorem 3.1. Let $\boldsymbol{S}$ be a commutative semigroup and let $T \in Mul_l(\boldsymbol{S})$ be injective . Then $\ell^1(\boldsymbol{S})$ is semisimple if and only if $\ell^1(\boldsymbol{S_T})$ is semisimple.

Proof. Take $r, s \in \boldsymbol{S}$. Then $r^2 = s^2 = rs$ if and only if $T(r^2) = T(s^2) = T(r)T(s)$ or equivalently $r_0 r = s_0 s = r_0 s$, because T is injective**.** So, by theorem 5.8 [**8**], $\ell^1(\boldsymbol{S})$ is semisimple if and only if $\ell^1(\boldsymbol{S}_T)$ is semisimple.

Theorem 3.2. Let $\boldsymbol{S}$ be a discrete semigroup and $T \in Mul_l(\boldsymbol{S})$. Then
(i)The left multiplier T has an extension $\tilde{T} \epsilon Mul_l\left(\ell^1(\boldsymbol{S})\right)$ with the norm decreasing.

(ii) The left multiplier T is injective if and only if so is $\tilde{T}$.
(iii) If T is injective then $\tilde{T}$ is an isometry and also $\ell^1(\boldsymbol{S}_T)$ and $\left(\ell^1(\boldsymbol{S})\right)_T$ are isomorphic.

Proof. (i) An arbitrary element $f \epsilon\, \ell^1(\boldsymbol{S})$ is of the form $f: \boldsymbol{S} \to \mathbb{C}$ such that $f(x) = 0$ except at the most countable subset A of **S**. If A is a finite subset of **S** then $f = \sum_{k=1}^{n} f(x_k)\, \delta_{x_k}$for some fixed $n \epsilon \mathbb{N}$. So in general we have

$$f = \sum_{x \epsilon S} f(x) \delta_x = \sum_{x \epsilon A} f(x) \delta_x = \sum_{k=1}^{\infty} f(x_k)\, \delta_{x_k}.$$

Now, for each $n \epsilon \boldsymbol{N}$, let $f_n = \sum_{k=1}^{n} f(x_k) \delta_{x_k}$ and define $\tilde{T}: \ell^1(\boldsymbol{S}) \longrightarrow \ell^1(\boldsymbol{S})$ by

$$\tilde{T}(\delta_x) = \delta_{T(x)} \qquad (x \epsilon \boldsymbol{S}),$$

$$\tilde{T}(f_n) = \sum_{k=1}^{\mathrm{n}} f(x_k)\, \tilde{T}\left(\delta_{x_k}\right) = \breve{f_n} \ .$$

For each $m, n \epsilon \boldsymbol{N}$ where $n \geq m$, we have

$$\left\|\tilde{T}(f_n) - \tilde{T}(f_m)\right\|_1 = \left\|\widetilde{f_n} - \widetilde{f_m}\right\|_1 = \left\|\sum_{k=m}^{k=n} f(x_k)\ \tilde{T}\left(\delta_{x_k}\right)\right\| = \left\|\sum_{k=m}^{k=n} f(x_k)\, \delta_{T(x_k)}\right\|$$
$$\leq \sum_{k=m}^{k=n} |f(x_k)| = \|f_n - f_m\|_1 .$$

So $\{\tilde{T}(f_n)\}_n$ is a Cauchy sequence and it is convergent. Now, we define $\tilde{T}(f) = \lim_n \tilde{f_n}$. Then the definition is well defined. Hence

$$\tilde{T}(f) = \sum_{k=1}^{\infty} f(x_k)\,\tilde{T}\left(\delta_{x_k}\right) = \tilde{f}\ ,$$

also

$$\left\|\tilde{f}\right\|_1 \leq \sum_{x_k \epsilon A} |f(x_k)| = \|f\|_1 \ \text{ or } \ \left\|\tilde{T}(f)\right\|_1 \leq \|f\|_1.$$

It shows that $\tilde{T}$ is norm decreasing.

In the following, we extend $\tilde{T}$ by linearity. Let $f, g\epsilon\ \ell^1(\boldsymbol{S})$. Then there are two at most countable sub set A, B of **S** such that

$$f = \sum_{x\epsilon A} f(x)\delta_x\ ,\ \ g = \sum_{x\epsilon B} g(x)\delta_x\ .$$

Suppose that $D = A \cup B$. So we have $f + g = \sum_{x\epsilon D} (f(x) + g(x))\delta_x$.

Then, it follows that

$$\tilde{T}(f+g) = \widetilde{f+g} = \sum_{x\epsilon D}(f(x)+g(x))\tilde{T}(\delta_x) = \sum_{x\epsilon A} f(x)\,\tilde{T}(\delta_x) + \sum_{x\epsilon B} g(x)\,\tilde{T}(\delta_x)$$
$$= \tilde{f} + \tilde{g}\ .$$

Also, if $\alpha\epsilon\mathbb{C}$,we have

$$\tilde{T}(\alpha f) = \widetilde{\alpha f} = \sum_{x\epsilon A} \alpha f(x)\tilde{T}(\delta_x) = \alpha \sum_{x\epsilon A} f(s)\tilde{T}(\delta_x) = \alpha\tilde{T}(f).$$

Therefore, $\tilde{T}$ is a bounded linear isometry.

Now, we prove that $\tilde{T}\epsilon Mul_l\left(\ \ell^1(\boldsymbol{S})\right)$. Take $x, y\epsilon\boldsymbol{S}$. Then

$$\tilde{T}\left(\delta_x * \delta_y\right) = \tilde{T}\left(\delta_{xy}\right) = \delta_{T(xy)} = \delta_{T(x)y} = \delta_{T(x)} * \delta_y = \tilde{T}(\delta_x) * \delta_y.$$

Let $y\epsilon\boldsymbol{S}$ be fixed and $f, g\epsilon\ \ell^1(\boldsymbol{S})$. Then

$$\tilde{T}\left(f * \delta_y\right) = \tilde{T}\Big(\sum_{x\epsilon A} f(x)\,\delta_{xy}\Big) = \sum_{x\epsilon A} f(x)\tilde{T}\left(\delta_{xy}\right)$$
$$= \left(\sum_{x\epsilon A}\tilde{T}\ (\delta_x)\right) * \delta_y = \tilde{f} * \delta_y = \tilde{T}(f) * \delta_y.$$

In the general case, we have

$$\tilde{T}(f * g) = \tilde{T}\Big(\sum_{x\epsilon A} f(x)\left(\sum_{y\epsilon B} g(y)\right)\delta_{xy}\Big) = \sum_{x\epsilon A} f(x)\sum_{y\epsilon B}\ g(y)\tilde{T}(\delta_x) * \delta_y$$
$$= \sum_{x\epsilon A} f(x)\tilde{T}(\delta_x) * \sum_{y\epsilon B}\ g(y)\,\delta_y = \tilde{T}(f) * g\ .$$

This shows that $\tilde{T}$ is a multiplier on $\ell^1(\boldsymbol{S})$.

(ii) Let T be injective. Take $x, y\epsilon\boldsymbol{S}$ and suppose that $\tilde{T}(\delta_x) = \tilde{T}\left(\delta_y\right)$. Then $\delta_{T(x)} = \tilde{T}(\delta_x) = \tilde{T}\left(\delta_y\right) = \delta_{T(y)}$.

Therefore, $T(x) = T(y)$. Since T is injective, we have $x = y$. It follows that $\delta_x = \delta_y$, consequently $\tilde{T}$ is injective.

Conversely, the same argument shows that the converse holds.

(iii) Let T be injective and $f\epsilon\ell^1(\boldsymbol{S})$. Then there exists at most a countable subset $A \subseteq \boldsymbol{S}$ such that

$$f = \sum_{x \in A} f(x)\delta_x$$

Since A and $T(A)$ have the same cardinal number, $\|\tilde{T}(f)\| = \|\sum_{x\epsilon A} f(x)\,\delta_x\| = \sum_{x\epsilon A}|f(x)| = \|f\|_1$, so $\tilde{T}$ is an isometry.

Now, we can define a new multiplication " $\boxed{*}$ " on $\ell^1(\boldsymbol{S})$ as follow

$$f \boxed{*}\, g = f * \tilde{T}g \qquad (f, g\epsilon \ell^1(\boldsymbol{S})) .$$

By a similar argument in theorem1.31 $[\mathbf{10}]$, $\ell^1(\boldsymbol{S})$ with the new product is a Banach algebra that is denoted it by $\ell^1(\boldsymbol{S})_{\tilde{T}}$. We define the map $\Psi: \ell^1(\boldsymbol{S}_T) \longrightarrow \ell^1(\boldsymbol{S})_{\tilde{T}}$, by

$$\Psi(\delta_x) = \delta_x \qquad (x\epsilon \boldsymbol{S}).$$

Take $x, y\, \epsilon \boldsymbol{S}$. Then

$$\begin{aligned}\Psi(\delta_x * \delta_y) &= \Psi(\delta_{x \circ y}) = \delta_{xT(y)} = \delta_x * \delta_{T(y)} \\ &= \delta_x * \tilde{T}(\delta_y) = \delta_x \boxed{*}\, \delta_y \\ &= \Psi(\delta_x) \boxed{*}\, \Psi(\delta_y).\end{aligned}$$

So, in general case, we have

$$\Psi(f * g) = \Psi\,(f) \boxed{*}\, \Psi(g) \qquad (f, g\epsilon \ell^1(\boldsymbol{S})) .$$

Thus, Ψ is an isomorphism. Therefore $\ell^1(\boldsymbol{S}_{\boldsymbol{T}})$ and $\ell^1(\boldsymbol{S})_{\tilde{T}}$ are isomorphic

Theorem 3.3. Let **S** be a semigroup and $T \in Mul_l(\boldsymbol{S})$ be bijective. Then $\ell^1(\boldsymbol{S})$ is amenable if and only if $\ell^1(\boldsymbol{S}_T)$ is amenable.

Proof. By theorem 3.2, we have $\ell^1(\boldsymbol{S}_{\boldsymbol{T}}) \cong \ell^1(\boldsymbol{S})_{\tilde{T}}$. Suppose that $\ell^1(\boldsymbol{S}_T)$ is amenable and define $\varphi: \ell^1(\boldsymbol{S})_{\tilde{T}} \longrightarrow \ell^1(\boldsymbol{S})$ by $\varphi(f) = \tilde{T}(f)$. Take $x, y\epsilon \boldsymbol{S}$. Then

$$\begin{aligned}\varphi(\delta_x \boxed{*}\, \delta_y) &= \tilde{T}(\delta_x \boxed{*}\, \delta_y) = \tilde{T}(\delta_{xT(y)}) = \tilde{T}(\delta_x * \delta_{T(y)}) = \tilde{T}(\delta_x) * \delta_{T(y)} \\ &= \tilde{T}(\delta_x) * \tilde{T}(\delta_y) = \varphi(\delta_x) * \varphi(\delta_y) .\end{aligned}$$

Now, by induction and continuity of $\tilde{T}$, we have

$$\varphi(f \boxed{*}\, g) = \varphi(f) * \varphi(g) .$$

If T is bijective, $\tilde{T}$ is bijective. Therefore φ is an epimorphism of $\ell^1(\boldsymbol{S}_{\boldsymbol{T}})$ onto $\ell^1(\boldsymbol{S})$.

Hence , by proposition 2.3.1 $[\mathbf{16}]$ $\ell^1(\boldsymbol{S})$ is amenable.

Conversely, suppose that $\ell^1(\boldsymbol{S})$ is amenable. Since T is bijective, $\tilde{T}$ is bijective. Therefore $\tilde{T}^{-1}$exists. Now define $\theta: \ell^1(\boldsymbol{S}) \longrightarrow \ell^1(\boldsymbol{S}_{\boldsymbol{T}})\ [\cong \ell^1(\boldsymbol{S})_{\tilde{T}}]$ by $\theta(f) = \tilde{T}^{-1}(f)$. Take $x, y\, \epsilon \boldsymbol{S}$ .Then

$$\begin{aligned}\theta(\delta_x * \delta_y) &= \tilde{T}^{-1}(\delta_{xy}) = \tilde{T}^{-1}(\delta_x)\tilde{T}\tilde{T}^{-1}(\delta_y) = \tilde{T}^{-1}(\delta_x) \boxed{*}\, \tilde{T}^{-1}(\delta_y) \\ &= \theta(\delta_x) \boxed{*}\, \theta(\delta_y) .\end{aligned}$$

Similarly θ is an epimorphism from $\ell^1(\boldsymbol{S})$ onto $\ell^1(\boldsymbol{S}_T)$. By proposition 2.3.1 [**16**] $\ell^1(\boldsymbol{S}_T)$ is amenable.

Note that, in general, it is not known when $\ell^1(\boldsymbol{S})$ is weakly amenable. For more detials see [**2**].

Theorem3.4. Let **S** be a semigroup and $T \in Mul_l(\boldsymbol{S})$ be bijective . Then, if **S** is completely regular then $\ell^1(\boldsymbol{S}_T)$ is weakly amenable.

Proof. It is enough to prove that $\boldsymbol{S}_T$ is completely regular, then by theorem 3.6 [**2**], $\ell^1(\boldsymbol{S_T})$ can be weakly amenable. Take $s\epsilon\boldsymbol{S}$. Then there exists $r\epsilon\boldsymbol{S}$ such that $T(s) = T(s)T(r)T(s)$, $T(r)T(s) = T(s)T(r)$, since T is bijective and **S** $= T(\boldsymbol{S})$ is completely regular. So we have $T(s) = T(s \circ r \circ s)$ and $T(r \circ s) = T(s \circ r)$. Hence $s = s \circ r \circ s$ and $r \circ s = s \circ r$ for some $r\epsilon\boldsymbol{S}$, since T is injective. Therefore $\boldsymbol{S}_T$ is completely regular.

Corollary.3.5. Suppose that **S** is a commutative completely regular semigroup and $T \in Mul_l(\boldsymbol{S})$ is injective. Then $\ell^1(T(\boldsymbol{S})_T)$ is weakly amenable.

Proof. [2, theorem 3.6] $\ell^1(\boldsymbol{S})$ is weakly amenable. Define $\varphi: \boldsymbol{S} \longrightarrow \ell^1(\boldsymbol{S})_T$ by

$$\varphi(s) = T^{-1}(s) \qquad (s\epsilon\boldsymbol{S}).$$

We show that φ is a homomorphism . Take $s\epsilon\boldsymbol{S}$, then we have

$$\varphi(rs) = T^{-1}(rs) = T^{-1}(r)s = T^{-1}(r) \circ (T^{-1}s) .$$

So φ is a homomorphism. Then by proposition 2.1[**7**], $\ell^1(T(\boldsymbol{S})_T)$ is weakly amenable. In the case that $\boldsymbol{S}$ is a group, it is easy to see that the amenability of $\boldsymbol{S}$ implies the amenability of $\ell^1(\boldsymbol{S}_T)$. Indeed, when $\boldsymbol{S}$ is a group, by theorem 2.1, $\boldsymbol{S}_T$ is a semigroup and one can easily prove that S_T is also a group. On the other hand, $Mul_l(\boldsymbol{S}) \cong \boldsymbol{S}$ because $\boldsymbol{S}$ is a unital semigroup, so each $T\epsilon Mul_l(\boldsymbol{S})$ is inner and of the form $T = L_s$ for some $s\epsilon\boldsymbol{S}$. Also $T^{-1} = L_{a^{-1}}$ exists, since $\boldsymbol{S}$ is a group. Then the map $\theta: \boldsymbol{S}_T \to \boldsymbol{S}$ defined by $\theta(s) = T(s)$ is an isomorphism; that is $\boldsymbol{S} \cong \boldsymbol{S}_T$. Thus we have the following result:

Corollary 3.6. Let $\boldsymbol{S}$ be a cancellative regular discrete semigroup. Then $\ell^1(\boldsymbol{S})$ is amenable if and only if $\ell^1(\boldsymbol{S}_T)$ is amenable.

Proof. By [**9**, Exercise 2.6.11] $\boldsymbol{S}$ is a group. So the assertion holds by [15, theorem 2.1.8]

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Examples

In this section we present some examples which either comments on our results or indicate necessary condition in our theorems.

4.1. There are semigroups $\boldsymbol{S}$ and $T \in Mul_l(\boldsymbol{S})$ such that the background semigroups $\boldsymbol{S}$ are not commutative but their induced semigroups $\boldsymbol{S}_T$ are commutative.

This example shows that the condition $\overline{T(\boldsymbol{S})} = \boldsymbol{S}$, in theorem 2.2, can not be omitted.

Let $\boldsymbol{S}$ be the set {a, b, c, d, e} with operation table given by

.	a	b	c	d	e
a	a	a	a	d	d
b	a	b	c	d	d
c	a	c	b	d	d
d	d	d	d	a	a
e	d	e	e	a	a

Clearly$(\boldsymbol{S},.)$ is a non-commutative semigroup. Now, put $T = L_a$ where $L_a(x) = ax$ for all $x\varepsilon\boldsymbol{S}$. One can get easily the operation table of $\boldsymbol{S}_T$ as fallow:

∘	a	b	c	d	e
a	a	a	a	d	d
b	a	a	a	d	d
c	a	a	a	d	d
d	d	d	d	a	a
e	d	d	d	a	a

The operation table shows that the induced semigroup $\boldsymbol{S}_T$ is commutative and $T(\boldsymbol{S}) \neq \boldsymbol{S}$. Also the other induced semigroup $\boldsymbol{S}_T$ is commutative for $T = L_d$ analogously.

Now we present some important theorems from[**14**] that we need in the following examples:

Theorem 4. 2. Let $\boldsymbol{S}$ be a semigroup. Suppose that $\ell^1(\boldsymbol{S})$ is amenable. Then

(i) $\boldsymbol{S}$ is amenable

(ii) $\boldsymbol{S}$ is regular.

(iii) $E(\boldsymbol{S})$ is finite.

(iv) $\ell^1(\boldsymbol{S})$has an identity.

Proof. (i) That is lemma 3 in [**5**].

(ii) and (iii) See theorem 2 in[**6**].

(iv) That is corollary 10.6 in[**4**].

Theorem 4.3. Let $\boldsymbol{S}$ be a finite semigroup. Then the following statements are equivalent:

(i) $\ell^1(\boldsymbol{S})$ is amenable.

(ii) $\boldsymbol{S}$ is regular and $\ell^1(\boldsymbol{S})$ is nuital.

(ii)) $\boldsymbol{S}$ is regular and $\ell^1(\boldsymbol{S})$ is semisimple.

Proof. Refer to [**3**].

4.4. There are semigroups $\boldsymbol{S}$ and $T \in Mul_l(\boldsymbol{S})$ such that $\boldsymbol{S}$ and $\ell^1(\boldsymbol{S})$ are amenable but $\boldsymbol{S}_T$ is not regular and also, $\ell^1(\boldsymbol{S}_T)$ is not amenable.

This example shows that two semigroup algebras $\ell^1(\boldsymbol{S})$ and $\ell^1(\boldsymbol{S}_T)$ can be different in some properties. Also, it notifies that the bijectivity of T in the theorem 3.3 is essential. Put $\mathbf{S} = \{x_0, x_1, x_2, \dots, x_n\}$ with the operation $x_i x_j = x_{Max\{i,j\}}$ $(0 \le i, j \le n,$ $n \ge 2$).

Then **S** is a semigroup. Since

$$Max\{i, Max\{j,k\}\} = Max\{Max\{i,j\}, k\} = Max\{i,j,k\}.$$

We denote it by $\boldsymbol{S}_\vee$. This semigroup is commutative. So by (0.18) in [**12**], it is amenable. $\boldsymbol{S}_\vee$ is a unital semigroup and has a zero; indeed, $e_s = x_0$ and $o_s = x_n$. Also, it is a regular semigroup and $Mul(\boldsymbol{S}_\vee) \cong \boldsymbol{S}_\vee$ because $\boldsymbol{S}_\vee$ has an identity.

Evidently, $\boldsymbol{S}_\vee$ is regular since each $s \in \boldsymbol{S}_\vee$ is idempotent. The semigroup algebra $\ell^1(\boldsymbol{S}_\vee)$ is a unital algebra because $\boldsymbol{S}_\vee$ has an identity. So by theorem 4.3 (ii) $\ell^1(\boldsymbol{S}_\vee)$ is amenable.

Now, take $T = L_{x_k}$ for a fixed $x_k \epsilon \boldsymbol{S}$ where $k \ge 1$. By theorem 2.2, $(\boldsymbol{S}_\vee)_T$ is commutative so is amenable. We show that T is neither injective and nor surjective.

Take $x_i \epsilon \boldsymbol{S}_\vee$, then $T x_i = x_k x_i = x_{max\{k,i\}}$. So

$$\mathrm{T}(\boldsymbol{S}_\vee) = \{x_k, x_{k+1}, \dots, x_n\} \ne \boldsymbol{S}_\vee.$$

Hence, T is not surjective.

Again, take distinct elements x_i, x_j in $\boldsymbol{S}_\vee$ for some $i, j < k$ such that $T(x_i) = T(x_j)$. Then we have $x_{max\{k,i\}} = x_{max\{k,j\}}$ but $x_i \ne x_j$. So T is not injective.

We prove that $(\boldsymbol{S}_\vee)_T$ is not regular. If $(\boldsymbol{S}_\vee)_T$ is regular, then for $x_{k-1} \epsilon \boldsymbol{S}_\vee$ there exists an element $x_j \epsilon \boldsymbol{S}_\vee$ such that

$$x_{k-1} = x_{k-1} \,{}_{0}\, x_j \,{}_{0}\, x_{k-1} = x_{Max\{k,j\}} .$$

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That implies that $max\{k,j\} = k-1$; which is impossible. Consequently, by theorem 4.2 (ii) or 4.3 (ii), $\ell^1((\boldsymbol{S}_\vee)_T)$ is not amenable.

Also, the inequality $\boldsymbol{S}_\vee \circ \boldsymbol{S}_\vee = \{x_k, x_{k+1}, \dots, x_n\} \neq \boldsymbol{S}_\vee$ shows that $\ell^1((\boldsymbol{S}_\vee)_T)$ is not weakly amenable. In the next example we show that in the theorem 3.2 (iii) the condition "injectivity of " can not be omitted.

4.5 There are a semigroup $\boldsymbol{S}$ and $T \in Mul_l(\boldsymbol{S})$ such that $T \in Mul_l(\boldsymbol{S})$is not injecyive and the corresponding $\tilde{T} \in Mul_l\left(\ell^1(\boldsymbol{S}_T)\right)$ is not an isometry.

Suppose that $\boldsymbol{S}_\vee$ is a semigroup as in example 4.4 and $T = L_{x_k}$for some fixed $1 < k < n$. If $f \epsilon \ell^1(\boldsymbol{S}_\vee)$ then $f = \sum_{i=0}^n f(x_i)\delta_{x_i}$ and also $\tilde{T}(f) = \sum_{i=0}^n f(x_i)\delta_{T(x_i)}$. But

$$T(x_i) = \begin{cases} x_i & k < i \leq n \\ x_k & 0 \leq i \leq k \end{cases},$$

so

$$\tilde{T}(f) = \left(\sum_{i=0}^k f(x_i)\right) \delta_{x_k} + \sum_{i=k+1}^n f(x_i)\delta_{T(x_i)} .$$

Hence

$$\begin{aligned} \|\tilde{T}(f)\| &= \left|\sum_{i=0}^k f(x_i)\right| + \sum_{i=k+1}^n |f(x_i)| \\ &\leq \sum_{i=0}^k |f(x_i)| + \sum_{i=k+1}^n |f(x_i)| = \|f\|_1, \end{aligned}$$

It shows that $\tilde{T}$ is not an isometry.

4.6. There are semigroups $\boldsymbol{S}$ and $T \in Mul_l(\boldsymbol{S})$ such that $\ell^1(\boldsymbol{S})$ is semisimple. But $\ell^1(\boldsymbol{S}_T)$ is not semisimple. This example remind that, in theorem 3.1 the multiplier T must be injective.

Let **S** be a set $\{x_0, x_1, \dots, x_n\}$ where $n\epsilon\boldsymbol{N}$ and $n \geq 3$ is fixed. by operation given by $xy = x_{min\{i,j\}}$, **S** is a commutative semigroup. Since

$$min\{i, min\{j,k\}\} = min\{min\{i,j\},k\} = min\{i,j,k\} \qquad (i,j,k\epsilon\boldsymbol{N}).$$

We denote it briefly by $\boldsymbol{S}_\wedge$ For each $x, y\epsilon\boldsymbol{S}$ the equality $x^2 = y^2 = xy$ implies $x = y$. So by Theorem 5.8 [**8**] $\ell^1(\boldsymbol{S}_\wedge)$ is semisimple.

Now, let $T = L_{x_k}$ for a fixed $1 \leq k < n-1$. It is easy to see that $T(x_k) = T(x_n)$ but $x_k \neq x_n$. So the multiplier T is not injective.

We show that neither $\boldsymbol{S}_\wedge$ nor $\ell^1(\boldsymbol{S}_\wedge)_T$ is semisimple.

Each ideal of $\boldsymbol{S}$ is of the form

$$\boldsymbol{I}_m = \{x_0, x_1, \dots, x_m\} \qquad (m \leq n).$$

We claim that $\boldsymbol{S_T}$ is not semisimple. Since for each $m\epsilon\boldsymbol{N}$ we have

$$\boldsymbol{I}_m \circ \boldsymbol{I}_m = \begin{cases} \boldsymbol{I}_m & m \le k \\ \boldsymbol{I}_k & m > k \end{cases} .$$

On the other hand, for each $x_i, x_j \epsilon \boldsymbol{S}$ where $i \neq j$ and $i, j > k$, we have $x_i \circ x_i = x_j \circ x_j = x_i \circ x_j = x_k$, while $x_i \neq x_j$. Thus, Theorem 5.8 [**8**] shows that $\ell^1(\mathbf{S}_\wedge)_T$ is not semisimple .

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