

A Nonholonomic Mechanical Structure for the Two-Dimensional Monolayer Systems

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Received: 26 Dec 2014

Revised: 17 Sep 2017

Abstract

In this paper we study the dynamics of the 2D-motion of a particle of monolayer. A geometric approach to nonholonomic constrained mechanical systems is applied to a problem from the two dimensional geometric dynamics of the Langmuir-Blodgett monolayer. We consider a constraint submanifold of the 1-jet space $J^1(T, R^2)$, corresponding to the given constraint condition in a monolayer space and construct the corresponding constrained mechanical system on the constraint submanifold. Then the equation of motion defined on the constraint submanifold is presented.

Keywords. Nonholonomic constraints, 2D-monolayer Lagrangian.

Mathematical Classification subject: 53C80, 70G45.

Introduction

Physical systems are often subjected to various types of constraints. Generally, these constraints are position, or geometric (holonomic), constraints, or velocity (nonholonomic) constraints. If a system has constraint equations that involve velocities or derivatives of system coordinates, the constraint equations are said to be nonholonomic and the mechanical system is said to be a nonholonomic system. Although almost all the work on nonholonomic systems is concerned with the case of constraints linear in components of velocities, but a geometric theory covering general nonholonomic systems has been developed by Krupkova [2], [3].

An example of such a system might be monolayer systems. In this paper we present geometric concept of the theory of nonholonomic mechanical systems developed by

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Krupkova [3] and consider the application of this theory to a problem from the two dimensional geometric dynamics of the Langmuir-Blodgett monolayer [4]. To this aim, we consider a constraint submanifold of the 1-jet space $J^1(T, R^2)$ corresponding to the given constraint conditions and construct the corresponding constrained mechanical system on the constraint submanifold. Then the equations of motion defined on the constraint submanifold are presented.

Section 3 of this paper contains a brief outline of basic geometrical concepts of nonholonomic mechanical systems that we will need. For proofs and more details see [2], [3].

Nonholonomic mechanical systems in a monolayer space

In this section we begin with a brief introduction to a monolayer space [1], [5] and then describe the nonholonomic mechanical system related to it. To obtain the 2D-motion equation of a particle of monolayer, we define a first order mechanical system $[\alpha]$ in this space and calculate the nonholonomic constrained system $[\alpha_Q]$ related to the mechanical system $[\alpha]$.

We start with the usual physical time defined by the Euclidian manifold $(T = [0, \infty))$ and we also consider the plane manifold R^2 having the polar coordinates (r, φ) , where $r > 0$ and $\varphi \in [0, 2\pi)$, and construct the 1-jet vector bundle $J^1(T, R^2) \rightarrow R \times R^2$, locally endowed with the coordinates $(t, q^1, q^2, \dot{q}^1, \dot{q}^2) := (t, r, \varphi, \dot{r}, \dot{\varphi})$.

Using the special function:

$$f(z) = - \int_{-z}^{\infty} \frac{e^{-t}}{t} dt$$

we study the 2D-motion of a particle of monolayer in plane moving in such a way, that the square of its speed decreases proportionally to the reciprocal value of time passed from the beginning of the motion.

We denote by (t) the coordinate on $X = T$, by (t, r, φ) fibered coordinates on $Y = T \times R^2$, and by $(t, r, \varphi, \dot{r}, \dot{\varphi})$ the associated coordinates on $J^1(T, R^2)$.

This particle of monolayer governed by the jet Lagrangian function $L: J^1(T, R^2) \rightarrow R$ defined by

$$L(t, r, \dot{r}, \dot{\varphi}) = \frac{m}{2} \dot{r}^2 + \frac{mr^2}{2} \dot{\varphi}^2 - \underbrace{pr^5 |V| e^{\frac{2|V|t}{r}} \cdot \dot{r}^{-1}}_{U_s(t,r)} + U(t, r), \quad (1)$$

where we have the following physical meanings: m is the *mass* of the particle; V is the *LB-monolayer compressing speed*; p is a *constant monolayer parameter* given by the physical formula:

$$p = \frac{\pi^2 q^2}{\varepsilon \varepsilon_0} \frac{\rho_0^2}{R_0^2};$$

$U_s(t, r)$ is an *electro capillarity potential energy* including the monomolecular layer function:

$$U(t, r) = p \left\{ \left[-\frac{4}{3} r^5 + \frac{16}{15} (|V|t) r^4 + \frac{1}{30} (|V|t)^2 r^3 + \frac{1}{45} (|V|t)^3 r^2 + \frac{1}{45} (|V|t)^4 r + \frac{2}{45} (|V|t)^5 \right] e^{\frac{2|V|t}{r}} - \frac{4}{45} + \frac{(|V|t)^6}{r} f\left(\frac{2|V|t}{r}\right) \right\}$$

According to the Lagrange theory on fibered manifolds [3], to every Lagrangian there exists a unique Lepagean equivalent, denoted by θ_λ , where λ is a Lagrangian on $J^1(T, R^2)$. The Euler-Lagrange form defined by the Lepagean equivalent depends only on the Lagrangian λ . Therefore it is referred to as the Euler-Lagrange form of the Lagrangian λ , and is denoted by E_λ . Now, we shall find chart expressions for the Lepagean and the Euler-Lagrange forms.

In fibered coordinates $(t, r, \varphi, r, \dot{\varphi})$ on $J^1(T, R^2)$, the module of contact 1-forms (denoted by $\Omega^1 J^1(T, R^2)$) is locally generated by the forms $\omega^1 = dr - r \cdot dt$ and $\omega^2 = d\varphi - \dot{\varphi} dt$. Put $\lambda = L dt$ and denote

$$\theta_\lambda = L dt + \frac{\partial L}{\partial \dot{r}} \omega^1 + \frac{\partial L}{\partial \dot{\varphi}} \omega^2, \tag{2}$$

accordingly

$$\begin{aligned} \theta_\lambda = & \left(\frac{1}{2} m (\dot{r}^2 + r^2 \dot{\varphi}^2) - p r^5 |V| e^{\frac{2|V|t}{r}} \dot{r}^{-1} + U(t, r) \right) dt \\ & + \left(m \dot{r} + p r^5 |V| e^{\frac{2|V|t}{r}} \dot{r}^{-2} \right) \omega^1 + (m r^2 \dot{\varphi}) \omega^2. \end{aligned} \tag{3}$$

We define a first order mechanical system $[\alpha]$ on the fibered manifold $J^1(T, R^2) \rightarrow R \times R^2$ represented by the 2-form with respect to (15)

$$\begin{aligned} \alpha = d\theta_\lambda + F = & \left(m r \dot{\varphi}^2 - \frac{5p r^4 |V| e^{\frac{2|V|t}{r}}}{\dot{r}} + \frac{2pr^3 |V|^2 t e^{\frac{2|V|t}{r}}}{\dot{r}} + \frac{\partial U}{\partial r} \right) dr \wedge dt \\ & + \left(\frac{2pr^4 |V|^2 e^{\frac{2|V|t}{r}}}{\dot{r}^2} \right) dt \wedge \omega^1 + \left(\frac{5pr^4 |V| e^{\frac{2|V|t}{r}}}{\dot{r}^2} - \frac{2pr^3 |V|^2 t e^{\frac{2|V|t}{r}}}{\dot{r}^2} \right) dr \wedge \omega^1 \\ & + \left(m - \frac{2pr^5 |V| e^{\frac{2|V|t}{r}}}{\dot{r}^3} \right) d\dot{r} \wedge \omega^1 + (2mr \dot{\varphi}) dr \wedge \omega^2 + (mr^2) d\dot{\varphi} \wedge \omega^2 + F. \end{aligned} \tag{4}$$

This mechanical system is related to the dynamical form with respect to (11)

$$E = E_1 dr \wedge dt + E_2 d\varphi \wedge dt, \quad (5)$$

where

$$E_1 = \left(mr\dot{\varphi} - \frac{10pr^4 |V| e^{\frac{2|V|t}{r}}}{\dot{r}} + \frac{4pt r^3 |V|^2 e^{\frac{2|V|t}{r}}}{\dot{r}} - \frac{2pr^4 |V|^2 e^{\frac{2|V|t}{r}}}{\dot{r}^2} + \frac{\partial U}{\partial r} \right) - \left(m + \frac{2pr^5 |V| e^{\frac{2|V|t}{r}}}{\dot{r}^2} \right) r\ddot{r},$$

and

$$E_2 = -(2mrr' + mr^2\varphi).$$

We consider the nonholonomic constraint Q given by the equation

$$f(t, r, \varphi, r, \dot{\varphi}) \equiv [(r')^2 + (\dot{\varphi})^2] - \frac{1}{t} = 0, \quad (6)$$

which means that the particle's speed decreases proportionally to $\frac{1}{\sqrt{t}}$. In a neighborhood of the submanifold Q

$$\text{rank} \left(\frac{\partial f}{\partial \dot{r}}, \frac{\partial f}{\partial \dot{\varphi}} \right) = \text{rank} (2\dot{r}, 2\dot{\varphi}) = 1. \quad (7)$$

Let $U \subset J^1(T, R^2)$ be the set of all points, where $\dot{\varphi} > 0$, and consider on U the adapted coordinates $(t, r, \varphi, r, \bar{f})$, where $\bar{f} = \dot{\varphi} - g$, $g = \sqrt{\frac{1}{t} - (\dot{r})^2}$ is the equation of the constraint (6) in normal form.

The constrained system $[\alpha_Q]$ related to the mechanical system $[\alpha]$ and the constraint Q is the equivalence class of the 2-form (with respect to (18))

$$\alpha_Q = A'_1 \omega^1 \wedge dt + B'_{1,1} \omega^1 \wedge d\dot{r} + \bar{F} + \phi_{(2)} \quad (8)$$

on Q , where \bar{F} is any 2-contact 2-form and $\phi_{(2)}$ is any constraint 2-form defined on Q . Calculating $\bar{L} = L \circ \iota$ and calculating $A'_1, B'_{1,1}$ by relationships (19), (20)

$$\bar{L}(t, r, \varphi, \dot{r}, \dot{\varphi}) = L \left(t, r, \varphi, \dot{r}, \sqrt{\frac{1}{t} - (\dot{r})^2} \right).$$

Then

$$\bar{L} = \frac{m}{2} \dot{r}^2 + \frac{mr^2}{2} \left(\frac{1}{t} - \dot{r}^2 \right) - \underbrace{pr^5 |V| e^{\frac{2|V|t}{r}} \cdot \dot{r}^{-1}}_{U_S(t,r)} + U(t, r),$$

and

$$A'_1 = mr g^2 + 2mrr\dot{r}^2 - \frac{m\dot{r}r^2}{2t^2} g^2 - \frac{10pr^4 t |V|^2 e^{\frac{2|V|t}{r}}}{\dot{r}} + \frac{4pr^3 t |V|^2 e^{\frac{2|V|t}{r}}}{\dot{r}} - \frac{2pr^4 |V|^2 e^{\frac{2|V|t}{r}}}{\dot{r}^2} + \frac{\partial U}{\partial r},$$

$$B'_{1,1} = m(r^2 - 1) - \frac{mr^2}{tg^2} + \frac{2pr^5 t |V| e^{\frac{2|V|t}{r}}}{r^3}.$$

Reduced equation of motion of the constrained system is as follows

$$[A'_1 + B'_{1,1} r'] \circ J^2 \bar{\gamma} = 0, \tag{9}$$

where $\bar{\gamma} = (t, r(t), \varphi(t))$ is a section satisfying the constraint equation $f \circ J^1 \gamma = 0$.

Lagrangian systems on fibered manifolds

Throughout this section we consider a fibered manifold $\pi: Y \rightarrow X$ with a one dimensional base space X and $(m + 1)$ -dimensional total space Y . We use jet prolongations $\pi_1 : J^1(X, Y) \rightarrow X$ and $\pi_2 : J^2(X, Y) \rightarrow X$ and jet projections $\pi_{1,0} : J^1(X, Y) \rightarrow Y$ and $\pi_{2,1} : J^2(X, Y) \rightarrow J^1(X, Y)$. Configuration space at a fixed time is represented by a fiber of the fibered manifold π and a corresponding phase space is then a fiber of the fibered manifold π_1 . Local fibered coordinates on Y are denoted by (t, q^σ) , where $1 \leq \sigma \leq m$. The associated coordinates on $J^1(X, Y)$ and $J^2(X, Y)$ are denoted by $(t, q^\sigma, \dot{q}^\sigma)$ and $(t, q^\sigma, \dot{q}^\sigma, \ddot{q}^\sigma)$, respectively. In calculations we use either a canonical basis of one forms on $J^1(X, Y)$, $(dt, dq^\sigma, d\dot{q}^\sigma)$, or a basis adapted to the contact structure

$$(dt, \omega^\sigma, d\dot{q}^\sigma),$$

where

$$\omega^\sigma = dq^\sigma - \dot{q}^\sigma dt, \quad 1 \leq \sigma \leq m.$$

Whenever possible, the summation convention is used. If $f(t, q^\sigma, \dot{q}^\sigma)$ is a function defined on an open set of $J^1(X, Y)$ we write

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial q^\sigma} \dot{q}^\sigma + \frac{\partial f}{\partial \dot{q}^\sigma} \ddot{q}^\sigma, \quad \frac{\bar{d}f}{dt} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial q^\sigma} \dot{q}^\sigma.$$

A differential form ρ is called *contact* if $J^1 \gamma^* \rho = 0$ for every section γ of π . A contact 2-form ρ is called *1-contact* if for every vertical vector field ξ , $i_\xi \rho_1$ is a horizontal; ρ is 2-contact if $i_\xi \rho_1$ is 1-contact. The operator assigning to ρ its 1-contact part is denoted by p_1 .

If λ is a Lagrangian on $J^1(X, Y)$, we denote by θ_λ its *Lepage equivalent* or *Cartan form* and E_λ its Euler-Lagrange form, respectively. Recall that $E_\lambda = p_1 d\theta_\lambda$. In fibered coordinates where $\lambda = L(t, q^\sigma, \dot{q}^\sigma) dt$, we have

$$\theta_\lambda = L dt + \frac{\partial L}{\partial \dot{q}^\sigma} \omega^\sigma, \tag{10}$$

and

$$E_\lambda = E_\sigma (L) \omega^\sigma \wedge dt, \quad (11)$$

where the components $E_\sigma (L) = \frac{\partial L}{\partial q^\sigma} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}^\sigma}$, are the *Euler-Lagrange expressions*.

Since the functions E_σ are affine in the second derivatives we write

$$E_\sigma = A_\sigma + B_{\sigma\nu} \dot{q}^\nu,$$

where

$$A_\sigma = \frac{\partial L}{\partial q^\sigma} - \frac{\partial^2 L}{\partial t \partial \dot{q}^\sigma} - \frac{\partial^2 L}{\partial q^\nu \partial \dot{q}^\sigma} \dot{q}^\nu, \quad B_{\sigma\nu} = -\frac{\partial^2 L}{\partial \dot{q}^\sigma \partial \dot{q}^\nu}. \quad (12)$$

A section γ of π is called a *path* of the Euler-Lagrange form E_λ if

$$E_\lambda \circ J^2 \gamma = 0. \quad (13)$$

In fibered coordinates this equation represents a system of m second-order ordinary differential equations

$$A_\sigma \left(t, \gamma^\nu, \frac{d\gamma^\nu}{dt} \right) + B_{\sigma\rho} \left(t, \gamma^\nu, \frac{d\gamma^\nu}{dt} \right) \frac{d^2 \gamma^\rho}{dt^2} = 0, \quad (14)$$

for components $\gamma^\nu(t)$ of a section γ , where $1 \leq \nu \leq m$. These equations are called *Euler-Lagrange equations* or *motion equations* and their solutions are called *paths*.

Euler-Lagrange equations (14) can be written in the form

$$J^1 \gamma^* i_\xi \alpha = 0,$$

where $\alpha = d\theta_\lambda + F$ is any 2-form defined on an open subset $W \subset J^1(X, Y)$, such that $p_1 \alpha = E_\lambda$, and F is a 2-contact 2-form. In fibered coordinates we have $F = F_{\sigma\nu} \omega^\sigma \wedge \omega^\nu$, where $F_{\sigma\nu}(t, q^\rho, \dot{q}^\rho)$ are arbitrary functions. Recall from [3] that the family of all such (local) 2-forms:

$$\alpha = d\theta_\lambda + F = A_\sigma \omega^\sigma \wedge dt + B_{\sigma\nu} \omega^\sigma \wedge dq^\nu + F \quad (15)$$

is called a *first order Lagrangian system*, and is denoted by $[\alpha]$.

A non-holonomic constrained mechanical system is defined on the $(2m+1-k)$ -dimensional constraint submanifold $Q \subset J^1(X, Y)$ fibered over Y and given by k equations

$$f^i(t, q^1, \dots, q^m, \dot{q}^1, \dots, \dot{q}^m) = 0, \quad 1 \leq i \leq k,$$

where

$$\text{rank} \left(\frac{\partial f^i}{\partial \dot{q}^\sigma} \right) = k, \quad (16)$$

or following [6], equivalently in an explicit form

$$\dot{q}^{m-k+i} = g^i(t, q^\sigma, \dot{q}^1, \dot{q}^2, \dots, \dot{q}^{m-k}), \quad 1 \leq i \leq k. \quad (17)$$

By a nonholonomic constrained system arising from the Lagrangian system $[\alpha]$ and constraint forms on the constraint submanifold Q , we mean the equivalence class $[\alpha_Q]$ on Q , where

$$\alpha_Q = \iota^* d\theta_\lambda + \bar{F} + \phi_{(2)},$$

where \bar{F} is any 2-contact $\pi_{1,0}$ horizontal 2-form and ϕ_2 is any constraint 2-form defined on Q , and ι is the canonical embedding of Q into $J^1(X, Y)$. The local form of $[\alpha_Q]$ is

$$\alpha_Q = \sum_{l=1}^{m-k} A'_l \omega^l \wedge dt + \sum_{l,s=1}^{m-k} B'_{ls} \omega^l \wedge d\dot{q}^s + \bar{F} + \phi_{(2)}, \tag{18}$$

where the components A'_l and $B'_{l,s}$ are given by

$$A'_l = \frac{\partial \bar{L}}{\partial q^l} + \frac{\partial \bar{L}}{\partial q^{m-k+i}} \frac{\partial g^i}{\partial \dot{q}^l} - \frac{\bar{d}c}{dt} \frac{\partial \bar{L}}{\partial \dot{q}^l} + \left(\frac{\partial L}{\partial \dot{q}^{m-k+j}} \right)_l \left[\frac{\bar{d}c}{dt} \left(\frac{\partial g^j}{\partial \dot{q}^l} \right) - \frac{\partial g^j}{\partial q^l} - \frac{\partial g^j}{\partial q^{m-k+i}} \frac{\partial g^i}{\partial \dot{q}^l} \right],$$

$$B'_{l,s} = -\frac{\partial^2 \bar{L}}{\partial \dot{q}^l \partial \dot{q}^s} + \left(\frac{\partial L}{\partial \dot{q}^{m-k+i}} \right)_l \frac{\partial^2 g^i}{\partial \dot{q}^l \partial \dot{q}^s},$$

$\bar{L} = L \circ \iota$, and

$$\frac{\bar{d}c}{dt} = \frac{\partial}{\partial t} + \dot{q}^s \frac{\partial}{\partial q^s} + g^i \frac{\partial}{\partial q^{m-k+i}}.$$

The equations of the motion of the constrained system $[\alpha_Q]$ in fibered coordinates take the form

$$\left(A'_l + \sum_{s=1}^{m-k} B'_{l,s} \ddot{q}^s \right) \circ J^2 \bar{\gamma} = 0. \tag{19}$$

for components $\gamma^1(t), \gamma^2(t), \dots, \gamma^{m-k}(t)$ of a Q -admissible section $\bar{\gamma}$ dependent on time t and parameters $q^{m-k+1}, q^{m-k+2}, \dots, q^m$, which have to be determined as functions $\gamma^{m-k+1}(t), \gamma^{m-k+2}(t), \dots, \gamma^m(t)$ from the equations (17) of the constraint

$$\frac{dq^{m-k+i}}{dt} = g^i \left(t, q^\sigma, \frac{dq^1}{dt}, \frac{dq^2}{dt}, \dots, \frac{dq^{m-k}}{dt} \right), \quad 1 \leq i \leq k.$$

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