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A Nonholonomic Mechanical Structure for the Two-Dimensional Monolayer Systems

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Abstract

In this paper we study the dynamics of the 2D-motion of a particle of monolayer. A geometric approach to nonholonomic constrained mechanical systems is applied to a problem from the two dimensional geometric dynamics of the Langmuir-Blodgett monolayer. We consider a constraint submanifold of the 1-jet space $J^1(T,R^2)$, corresponding to the given constraint condition in a monolayer space and construct the corresponding constrained mechanical system on the constraint submanifold. Then the equation of motion defined on the constraint submanifold is presented.

Keywords. Nonholonomic constraints, 2D-monolayer Lagrangian.

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Introduction

Physical systems are often subjected to various types of constraints. Generally, these constraints are position, or geometric (holonomic), constraints, or velocity (nonholonomic) constraints. If a system has constraint equations that involve velocities or derivatives of system coordinates, the constraint equations are said to be nonholonomic and the mechanical system is said to be a nonholonomic system. Although almost all the work on nonholonomic systems is concerned with the case of constraints linear in components of velocities, but a geometric theory covering general nonholonomic systems has been developed by Krupkova [2], [3].

An example of such a system might be monolayer systems. In this paper we present geometric concept of the theory of nonholonomic mechanical systems developed by

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Krupkova [3] and consider the application of this theory to a problem from the two dimensional geometric dynamics of the Langmuir-Blodgett monolayer [4]. To this aim, we consider a constraint submanifold of the 1-jet space $J^1(T, R^2)$ corresponding to the given constraint conditions and construct the corresponding constrained mechanical system on the constraint submanifold. Then the equations of motion defined on the constraint submanifold are presented.

Section 3 of this paper contains a brief outline of basic geometrical concepts of nonholonomic mechanical systems that we will need. For proofs and more details see [2], [3].

Nonholonomic mechanical systems in a monolayer space

In this section we begin with a brief introduction to a monolayer space [1], [5] and then describe the nonholonomic mechanical system related to it. To obtain the 2D-motionn equation of a particle of monolayer, we define a first order mechanical system $[\alpha]$ in this space and calculate the nonholonomic constrained system $[\alpha_Q]$ related to the mechanical system $[\alpha]$.

We start with the usual physical time defined by the Euclidian manifold
$(T = [0, \infty))$ and we also consider the plane manifold R^2 having the polar coordinates (r, φ), where $r > 0$ and $\varphi \in [\ 0{,}2\pi\ )$, and construct the 1-jet vector bundle $J^1(T, R^2) \to R \times\ R^2$, locally endowed with the coordinates $(t, q^1, q^2, \dot{q}^1, q^{\dot{2}}) := (t, r, \varphi, \dot{r}, \varphi)$.

Using the special function:

$$f(z) = -\int_{-z}^{\infty} \frac{e^{-t}}{t} dt$$

we study the 2D-motion of a particle of monolayer in plane moving in such a way, that the square of its speed decreases proportionally to the reciprocal value of time passed from the beginning of the motion.

We denote by (t) the coordinate on $X = T$, by $(t, r\ , \varphi)$ fibered coordinates on $Y = T \times R^2$, and by $(t\ , r, \varphi,\ r\dot{;} \varphi)$ the associated coordinates on $J^1(T, R^2)$.

This particle of monolayer governed by the jet Lagrangian function $L: J^1(T, R^2) \to R$ defined by

$$L(t, r, \dot{r}, \dot{\varphi}) = \frac{m}{2}\dot{r}^2 + \frac{mr^2}{2}\dot{\varphi}^2 \underbrace{-pr^5|V|e^{\frac{2|V|t}{r}} \cdot\ \dot{r}^{-1} + U(t,r)}_{U_{s(t,r)}}, \tag{1}$$

where we have the following physical meanings: m is the *mass* of the particle; V is the *LB-monolayer compressing speed*; p is a *constant monolayer parameter* given by the physical formula:

$$p = \frac{\pi^2 q^2}{\varepsilon\varepsilon_0} \frac{\rho_0^2}{R_0^2};$$

$U_s(t,r)$ is an *electro capillarity potential energy* including the monomolecular layer function:

$$U(t,r) = p\left\{\left[-\frac{4}{3}r^5 + \frac{16}{15}(|V|t)r^4 + \frac{1}{30}(|V|t)^2 r^3 + \frac{1}{45}(|V|t)^3 r^2 + \frac{1}{45}(|V|t)^4 r + \frac{2}{45}(|V|t)^5\right] e^{\frac{2|V|t}{r}} - \frac{4}{45} + \frac{(|V|t)^6}{r} f(\frac{2|V|t}{r}).\right\}$$

According to the Lagrange theory on fibered manifolds [3], to every Lagrangian there exists a unique Lepagean equivalent, denoted by θ_λ, where λ is a Lagrangian on $J^1(T,R^2)$. The Euler-Lagrange form defined by the Lepagean equivalent depends only on the Lagrangian λ. Therefore it is referred to as the Euler-Lagrange form of the Lagrangian λ, and is denoted by E_λ. Now, we shall find chart expressions for the Lepagean and the Euler-Lagrange forms.

In fibered coordinates $(t, r, \varphi, r^{\cdot}, \varphi^{\cdot})$ on $J^1(T,R^2)$, the module of contact 1-forms (denoted by $\Omega^1 J^1(T,R^2)$) is locally generated by the forms $\omega^1 = dr - r^{\cdot}dt$ and $\omega^2 = d\varphi - \varphi^{\cdot}dt$. Put $\lambda = L\,dt$ and denote

$$\theta_\lambda = L\,dt + \frac{\partial L}{\partial \dot r}\omega^1 + \frac{\partial L}{\partial \dot\varphi}\omega^2, \tag{2}$$

accordingly

$$\theta_\lambda = \left(\frac{1}{2} m(\dot r^2 + r^2\dot\varphi^2) - p\,r^5 |V| e^{\frac{2|V|t}{r}} \dot r^{-1} + U(t,r)\right) dt + \left(m\dot r + p\,r^5|V|e^{\frac{2|V|t}{r}}\dot r^{-2}\right)\omega^1 + (m r^2 \dot\varphi)\omega^2. \tag{3}$$

We define a first order mechanical system $[\alpha]$ on the fibered manifold $J^1(T,R^2) \to R \times R^2$ represented by the 2-form with respect to (15)

$$\alpha = d\theta_\lambda + F = \left(mr\dot\varphi^2 - \frac{5p\,r^4 |V| e^{\frac{2|V|t}{r}}}{\dot r} + \frac{2pr^3 |V|^2 t\, e^{\frac{2|V|t}{r}}}{\dot r} + \frac{\partial U}{\partial r}\right) dr \wedge dt + \left(\frac{2pr^4|V|^2 e^{\frac{2|V|t}{r}}}{\dot r^2}\right) dt\wedge\omega^1 + \left(\frac{5pr^4|V|e^{\frac{2|V|t}{r}}}{\dot r^2} - \frac{2pr^3|V|^2 te^{\frac{2|V|t}{r}}}{\dot r^2}\right) dr\wedge\omega^1 + \left(m - \frac{2pr^5|V|e^{\frac{2|V|t}{r}}}{\dot r^3}\right) d\dot r\wedge\omega^1 + (2mr\dot\varphi) dr\wedge\omega^2 + (mr^2)d\dot\varphi\wedge\omega^2 + F. \tag{4}$$

This mechanical system is related to the dynamical form with respect to (11)

$$E = E_1\, dr \wedge dt + E_2\, d\varphi \wedge dt, \tag{5}$$

where

$$E_1 = \left(mr\dot{\varphi} - \frac{10pr^4\,|V|e^{\frac{2|V|t}{r}}}{\dot{r}} + \frac{4\,pt\,r^3\,|V|^2\, e^{\frac{2|V|t}{r}}}{\dot{r}} - \frac{2pr^4\,|V|^2 e^{\frac{2|V|t}{r}}}{\dot{r}^2} + \frac{\partial U}{\partial r} \right)$$

$$- \left(m + \frac{2pr^5|V|e^{\frac{2|V|t}{r}}}{\dot{r}2} \right) \ddot{r},$$

and

$$E_2 = -(2mr\dot{r} + mr^2\varphi).$$

We consider the nonholonomic constraint Q given by the equation

$$f\,(t, r, \varphi, \dot{r}, \dot{\varphi}) \equiv [\,(\dot{r})^2 + (\dot{\varphi})^2] - \frac{1}{t} = 0, \tag{6}$$

which means that the particle's speed decreases proportionally to $\frac{1}{\sqrt{t}}$. In a neighborhood of the submanifold Q

$$rank\left(\frac{\partial f}{\partial \dot{r}}, \frac{\partial f}{\partial \dot{\varphi}}\right) = rank\,(2\dot{r}, 2\dot{\varphi}) = 1. \tag{7}$$

Let $U \subset J^1(T, R^2)$ be the set of all points, where $\dot{\varphi} > 0$, and consider on U the adapted coordinates $(t, r, \varphi, \dot{r}, \bar{\mathrm{f}})$, where $\bar{\mathrm{f}} = \dot{\varphi} - g$, $g = \sqrt{\frac{1}{t} - (\dot{r})^2}$ is the equation of the constraint (6) in normal form.

The constrained system $[\alpha_Q]$ related to the mechanical system $[\alpha]$ and the constraint Q is the equivalence class of the 2-form (with respect to (18))

$$\alpha_Q = A_1^{'}\, \omega^1 \wedge dt + B_{1,1}^{'}\, \omega^1 \wedge d\dot{r} + \bar{\mathrm{F}} + \phi_{(2)} \tag{8}$$

on Q, where $\bar{\mathrm{F}}$ is any 2-contact 2-form and $\phi_{(2)}$ is any constraint 2-form defined on Q. Calculating $\bar{\mathrm{L}} = L \circ \iota$ and calculating $A_1^{'}, B_{1,1}^{'}$ by relationships (19), (20)

$$\bar{\mathrm{L}}(t, r, \varphi, \dot{r}, \dot{\varphi}) = L\left(t, r, \varphi, \dot{r}, \sqrt{\frac{1}{t} - (\dot{r})^2}\right).$$

Then

$$\bar{\mathrm{L}} = \frac{m}{2}\dot{r}^2 + \frac{mr^2}{2}\left(\frac{1}{t} - \dot{r}^2\right) \underbrace{- pr^5|V|e^{\frac{2|V|t}{r}} \cdot \dot{r}^{-1} + U(t, r)}_{U_{s(t,r)}},$$

and

$$A_1^{'} = mrg^2 + 2mr\dot{r}^2 - \frac{m\dot{r}r^2}{2t^2}g^2 - \frac{10pr^4t|V|^2\, e^{\frac{2|V|t}{r}}}{\dot{r}} + \frac{4pr^3t|V|^2\, e^{\frac{2|V|t}{r}}}{\dot{r}} - \frac{2pr^4|V|^2 e^{\frac{2|V|t}{r}}}{\dot{r}^2} + \frac{\partial U}{\partial r},$$

$$B_{1,1}^{'} = m(r^2 - 1) - \frac{mr^2}{tg^2} + \frac{2pr^5 t|V|e^{\frac{2|V|t}{r}}}{\dot{r}^3}\ .$$

Reduced equation of motion of the constrained system is as follows

$$\left[A_1^{'} + B_{1,1}^{'}\,\dot{r}\right] \circ\ J^2\bar{\gamma} = 0, \tag{9}$$

where $\bar{\gamma} = (t, r(t), \varphi(t))$ is a section satisfying the constraint equation $f \circ J^1\gamma = 0$.

Lagrangian systems on fibered manifolds

Throughout this section we consider a fibered manifold $\pi: Y \to X$ with a one dimensional base space X and $(m+1)-$dimensional total space Y. We use jet prolongations $\pi_1 : J^1(X,Y) \to X$ and $\pi_2 : J^2(X,Y) \to X$ and jet projections $\pi_{1,0} : J^1(X,Y) \to Y$ and $\pi_{2,1}: J^2(X,Y) \to J^1(X,Y)$. Configuration space at a fixed time is represented by a fiber of the fibered manifold π and a corresponding phase space is then a fiber of the fibered manifold π_1. Local fibered coordinates on Y are denoted by (t, q^σ), where $1 \le \sigma \le m$. The associated coordinates on $J^1(X,Y)$ and $J^2(X,Y)$ are denoted by (t, q^σ, q^σ) and $(t, q^\sigma, \dot{q}^\sigma, q^\sigma)$, respectively. In calculations we use either a canonical basis of one forms on $J^1(X,Y)$, $(dt, dq^\sigma, d\dot{q}^\sigma)$, or a basis adapted to the contact structure

$$(dt, \omega^\sigma, d\dot{q}^\sigma),$$

where

$$\omega^\sigma = dq^\sigma - \dot{q}^\sigma dt, \qquad 1 \le \sigma \le m\,.$$

Whenever possible, the summation convention is used. If f $(t, q^\sigma, \dot{q}^\sigma)$ is a function defined on an open set of $J^1(X,Y)$ we write

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial q^\sigma}\dot{q}^\sigma + \frac{\partial f}{\partial \dot{q}^\sigma} q^\sigma\,, \qquad \frac{\bar{d}f}{\bar{d}t} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial q^\sigma}\dot{q}^\sigma.$$

A differential form ρ is called *contact* if $J^1\gamma^*\rho = 0$ for every section γ of π. A contact 2 -form ρ is called *1-contact* if for every vertical vector field ξ, $i_\xi\, \rho_1$ is a horizontal; ρ is 2 -contact if $i_\xi\, \rho_1$ is 1-contact. The operator assigning to ρ it's 1-contact part is denoted by p_1.

If λ is a Lagrangian on $J^1(X,Y)$, we denote by θ_λ its *Lepage equivalent* or *Cartan form* and E_λ its Euler-Lagrange form, respectively. Recall that $E_\lambda = p_1 d\theta_\lambda$. In fibered coordinates where $\lambda = L\,(t, q^\sigma, \dot{q}^\sigma)\,dt$, we have

$$\theta_\lambda = L\,dt + \frac{\partial L}{\partial \dot{q}^\sigma}\,\omega^\sigma, \tag{10}$$

and

$$E_\lambda = E_\sigma\ (L)\omega^\sigma \wedge dt, \tag{11}$$

where the components $E_\sigma\ (L) = \frac{\partial L}{\partial q^\sigma} - \frac{d}{dt}\ \frac{\partial L}{\partial \dot{q}^\sigma}$, are the *Euler-Lagrange expressions*. Since the functions E_σ are affine in the second derivatives we write

$$E_\sigma = A_\sigma + B_{\sigma\nu}\, \ddot{q}^\nu,$$

where

$$A_\sigma = \frac{\partial L}{\partial q^\sigma} - \frac{\partial^2 L}{\partial t\, \partial \dot{q}^\sigma} - \frac{\partial^2 L}{\partial q^\nu \partial \dot{q}^\sigma}\ \dot{q}^\nu, \qquad B_{\sigma\nu} = -\frac{\partial^2 L}{\partial \dot{q}^\sigma\, \partial \dot{q}^\nu}. \tag{12}$$

A section γ of π is called a *path* of the Euler-Lagrange form E_λ if

$$E_\lambda \circ J^2\gamma = 0. \tag{13}$$

In fibered coordinates this equation represents a system of m second-order ordinary differential equations

$$A_\sigma\left(t, \gamma^\nu, \frac{d\gamma^\nu}{dt}\right) + B_\sigma\rho\left(t, \gamma^\nu, \frac{d\gamma^\nu}{dt}\right)\frac{d^2\gamma^\rho}{dt^2} = 0, \tag{14}$$

for components $\gamma^\nu(t)$ of a section γ, where $1 \le \nu \le m$. These equations are called *Euler-Lagrange equations* or *motion equations* and their solutions are called *paths*. Euler-Lagrange equations (14) can be written in the form

$$J^1\ \gamma^* i_\xi\ \alpha = 0,$$

where $\alpha = d\theta_\lambda + F$ is any 2-form defined on an open subset $W \subset J^1(X,Y)$, such that $p_1\alpha = E_\lambda$, and F is a 2-contact 2-form. In fibered coordinates we have $F = F_{\sigma\nu}\omega^\sigma \wedge \omega^\nu$, where $F_{\sigma\nu}(t, q^\rho, q^\rho)$ are arbitrary functions. Recall from [3] that the family of all such (local) 2-forms:

$$\alpha = d\theta_\lambda + F = A_\sigma\omega^\sigma \wedge dt + B_{\sigma\nu}\omega^\sigma \wedge dq^\nu + F \tag{15}$$

is called a *first order Lagrangian system*, and is denoted by $[\alpha]$.

A non-holonomic constrained mechanical system is defined on the (2m+1-k)-dimensional constraint submanifold $Q \subset J^1(X,Y)$ fibered over Y and given by k equations

$$f^i\left(t, q^1, \ldots, q^m, q^1, \ldots, \dot{q}^m\right) = 0, \qquad 1 \le i \le k,$$

where

$$rank\left(\frac{\partial f^i}{\partial \dot{q}^\sigma}\right) = k, \tag{16}$$

or following [6], equivalently in an explicit form

$$\dot{q}^{m-k+i} = g^i(t, q^\sigma, \dot{q}^1, \dot{q}^2, \ldots, \dot{q}^{m-k}), \qquad 1 \le i \le k. \tag{17}$$

By a nonholonomic constrained system arising from the Lagrangian system $[\alpha]$ and constraint forms on the constraint submanifold Q, we mean the equivalence class $[\alpha_Q]$ on Q, where

$$\alpha_Q = \iota^* d\theta_\lambda + \bar{F} + \phi_{(2)},$$

where $\bar{F}$ is any 2-contact $\pi_{1,0}$ horizontal 2-form and ϕ_2 is any constraint 2-form defined on Q, and ι is the canonical embedding of Q into $J^1(X,Y)$. The local form of $[\alpha_Q]$ is

$$\alpha_Q = \sum_{l=1}^{m-k} A_l^{'}\, \omega^l \wedge dt + \sum_{l,s=1}^{m-k} B_{ls}^{'}\, \omega^l \wedge d\dot{q}^s + \bar{F} + \phi_{(2)}, \tag{18}$$

where the components $A_l^{'}$ and $B_{l,s}^{'}$ are given by

$$A_l^{'} = \frac{\partial \bar{L}}{\partial q^l} + \frac{\partial \bar{L}}{\partial q^{m-k+i}} \frac{\partial g^i}{\partial \dot{q}^l} - \frac{\bar{d}c}{dt}\frac{\partial \bar{L}}{\partial \dot{q}^l} + \left(\frac{\partial L}{\partial \dot{q}^{m-k+j}}\right)_\iota \left[\frac{\bar{d}c}{dt}\left(\frac{\partial g^j}{\partial \dot{q}^l}\right) - \frac{\partial g^j}{\partial q^l} - \frac{\partial g^j}{\partial q^{m-k+i}} \frac{\partial g^i}{\partial \dot{q}^l}\right],$$

$$B_{l,s}^{'} = -\frac{\partial^2 \bar{L}}{\partial \dot{q}^l \partial \dot{q}^s} + \left(\frac{\partial L}{\partial \dot{q}^{m-k+i}}\right)_\iota \frac{\partial^2 g^i}{\partial \dot{q}^l \partial \dot{q}^s},$$

$\bar{L} = L \circ \iota$, and

$$\frac{\bar{d}c}{dt} = \frac{\partial}{\partial t} + \dot{q}^s \frac{\partial}{\partial q^s} + g^i \frac{\partial}{\partial q^{m-k+i}}.$$

The equations of the motion of the constrained system $[\alpha_Q]$ in fibered coordinates take the form

$$\left(A_l^{'} + \sum_{s=1}^{m-k} B_{l,s}^{'}\, \ddot{q}^s\right) \circ J^2\bar{\gamma} = 0. \tag{19}$$

for components $\gamma^1(t), \gamma^2(t), \ldots, \gamma^{m-k}(t)$ of a $Q-$ admissible section $\bar{\gamma}$ dependent on time t and parameters $q^{m-k+1}, q^{m-k+2}, \ldots, q^m$, which have to be determined as functions $\gamma^{m-k+1}(t), \gamma^{m-k+2}(t), \ldots, \gamma^m(t)$ from the equations (17) of the constraint

$$\frac{dq^{m-k+i}}{dt} = g^i\left(t, q^\sigma, \frac{dq^1}{dt}, \frac{dq^2}{dt}, \ldots, \frac{dq^{m-k}}{dt}\right), \quad 1 \le i \le k.$$

References

1. Balan V., Neagu M., "Jet Single-Time Lagrange Geometry and Its Applications", John Wiley & Sons, Inc., Hoboken, New Jersey) 2011).

2. Krupkova O., "Mechanical systems with nonholonomic constraints", J. Math. Phys. 38 (1997) 5098-5126.

3. Krupkova O., "The Geometry of Ordinary Variational Equations", Lecture Notes in Math. 1678, Springer, Berlin) 1997).

4. Motschmann H., Mohwald H., "Langmuir-Blodgett Films, Handbook of Applied Surface and Colloid Chemistry", (Ed. Krister Holmberg), Ch. **28** (2001), 629-648.

5. Neagu M., Krylova N. G., Grushevskaya H. V., "Jet theoretical Yang-Mills energy in the geometric dynamics of 2D-monolayer", Journal of Mathematical Physics, Vol. **54**, 031508 (2013), 1-14; doi: 10.1063/1.4795715.

6. Arnold V. I., Kozlov V. V., Neishtadt A. I., "Mathematical aspects of classical and celestial mechanics", Encyclopedia of Math. Sciences.