Numerical solution of nonlinear Volterra-Fredholm integral equations of the first kind using alternative Legendre polynomials

Sohrab Bazm*[1], Alireza Hosseini[2]
1. Department of Mathematics, Faculty of Sciences, University of Maragheh, Iran
2. School of Mathematics, Statistics and Computer Science, College of Science, University of Tehran, Iran

Received: 07/08/2019 Accepted: 03/01/2020

Extended Abstract

Paper pages (443-466)

Introduction

Nonlinear Volterra-Fredholm integral equations arise in many problems of real life, as, for example, continuum mechanics, potential theory, geophysics, electricity and magnetism, antenna synthesis problem, communication theory, mathematical economics, population genetics, radiation, the particle transport problems of astrophysics and reactor theory, fluid mechanics, etc.

The purpose of this article is to propose a numerical method for solving the first kind nonlinear Volterra-Fredholm integral equations of the form

$$f(x)=\int_0^x k_1(x,t)G_1\big(u(t)\big)dt+\int_0^1 k_2(x,t)G_2\big(u(t)\big)dt\,,\quad x\in[0,1], \tag{1}$$

where u is an unknown real valued function and f and k_i, $i=1,2$, are given continuous functions defined, respectively on $I:=[0,1]$ and $I\times I$, and $G_i, i=1,2$, are polynomials of u with constant coefficients. For convenience, we assume that

$$\mathrm{G_i}\big(u(x)\big)=u^{p_i}(x),\qquad i=1,2, \tag{2}$$

where $p_i, i=1,2$, are positive integers, but the method can be easily extended and applied to any nonlinear integral equations of the form (1). So, we consider the following integral equation

$$f(x)=\int_0^x k_1(x,t)u^{p_1}(t)dt+\int_0^1 k_2(x,t)u^{p_2}(t)dt\,,\quad x\in[0,1]. \tag{3}$$

Several methods have been proposed to solve integral equations of the first kind. For example, discretization methods for solving the linear first kind Volterra integral equation

$$f(x)=\int_0^1 G(x,t)u(t)dt\,,\qquad x\in[0,T], \tag{4}$$

*Corresponding Author: sbazm@maragheh.ac.ir

are introduced in [1-7]. A direct method for solving (4) is described in [8] based on block-pulse functions and their operational matrix of integration. A modification of block-Pulse functions for the numerical solution of (4) is introduced and used in [9]. Brunner [10], Gladwin [3] and Gladwin and Jeltsch [11] have considered numerical schemes for the nonlinear Volterra integral equation of the first kind

$$f(x)=\int_0^1 G(x,t,u(t))dt, \qquad x\in[0,T]. \tag{5}$$

A class of discretization methods for the numerical solution of (5) is introduced in [12]. This class of methods has previously been considered for the linear equation (4) by Scott [13], and includes linear multistep methods, reducible quadrature, block-by-block methods and collocation. A Haar wavelet method for the numerical solution of a class of nonlinear Volterra integral equations of the first kind is given in [14]. Also, a survey of numerical methods for solving nonlinear integral equations have been presented in [15].

As noted in [8], integral equations of the first kind are inherently ill-posed problems, meaning that the solutions generally unstable, and small changes to the problem can make very large changes to the obtained answers. This ill-posedness makes numerical solutions very difficult, a small error may lead to an unbounded error. In [16-17] different regularization methods have been proposed to overcome with the difficulty of ill-posed problems. In this paper, to overcome the ill-posedness, by assuming that

$$k_1(x,x)\neq 0, \qquad x\in[0,1],$$

and differentiating (3) with respect to x, we get the following second kind Volterra-Fredholm integral equation

$$u^{p_1}(x)=h(x)+\int_0^x r_1(x,t)u^{p_1}(t)dt+\int_0^1 r_2(x,t)u^{p_2}(t)dt, \quad x\in[0,1],$$

where

$$r_i(x,t)=-\frac{\partial}{\partial x}k_i(x,t)/k_1(x,x), \qquad i=1,2,$$

and

$$h(x)=f'(x)/k_1(x,x).$$

Material and methods

We briefly describe some characteristics of the alternative Legendre polynomials (ALPs) and derive their operational matrices of integration and the product. Then, the collocation method together with the ALP operational matrices are used to reduce the solution of equation (3) to the solution of a nonlinear system of algebraic equations. The error analysis of the method is given and some numerical examples are considered to demonstrate the efficiency and accuracy of the method. In the numerical examples presented in the paper, we have solved the nonlinear system of algebraic equations using function *FindRoot* in *Mathematica* software, which uses Newton's method as the default method. Experiments were performed on a personal computer using a 2.50 GHz processor.

Results and discussion

The ALP operational matrices of integration and the product are derived. These matrices together with the collocation method are used to solve numerically problem (1) with assumption (2). This approach transformed the considered problem to a nonlinear system of algebraic equations with unknown ALP coefficients of the exact solution. As it is illustrated by the presented examples, the approximate solutions are close to the exact ones and high accuracy results can be achieved only using a small number of basis functions.

Conclusion

- The effort required to implement the method is very low, while the accuracy is high.
- As the numerical results show, a few number of basic functions are enough to obtain a high accuracy approximation of the solution.
- The main feature of the method is that it converts the problem under investigation into a system of algebraic equations which can be easily solved by using iterative methods.

Keywords: Nonlinear integral equations of the first kind, Volterra-Fredholm integral equations, Alternative Legendre polynomials, Operational matrix.