Multicolor Size-Ramsey Number of Paths

Ramin Javadi*, Meysam Miralaei;

Department of Mathematical Sciences, Isfahan University of Technology, Isfahan, Iran

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Extended Abstract

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Introduction

The size-Ramsey number of a graph F denoted by $\hat{r}(\mathrm{F,r})$ is the smallest integer m such that there is a graph G with m edges with this property that for any coloring of the edges of G with r colors, G contains a monochromatic copy of F. The investigation of the size-Ramsey numbers of graphs was initiated by Erdős, Faudree, Rousseau and Schelp in 1978. Since then, Size-Ramsey numbers have been studied with particular focus on the case of trees and bounded degree graphs.

Addressing a question posed by Erdős, Beck [2] proved that the size-Ramsey number of the path P_n is linear in n by means of a probabilistic construction. In fact, Beck's proof implies that $\hat{r}(P_n,2) \leq 900n$ and this upper bound was improved several times. Currently, the best known upper bound is due to Dudek and Prałat [4] which proved that $\hat{r}(P_n,2) \leq 74n$. On the other hand, the first nontrivial lower bound for $\hat{r}(P_n,2)$ was provided by Beck and his result was subsequently improved by Dudek and Prałat [3] who showed that $\hat{r}(P_n,2) \geq 5n/2 - O(1)$. The strongest known lower bound, $\hat{r}(P_n, 2) \geq (3.75 - O(1))\, n$, was proved recently by Bal and DeBiasio [1].

Let us now move to the multicolor version of the problem. Dudek and Prałat [3] proved that

$$\frac{(r+3)r}{4} n - O(r^2) < \hat{r}(P_n, r) < 33\, r 4^r\ n.$$

It follows that $\hat{r}(P_n, r)$ is linear in terms of n for any fixed value of r, however the two bounds are quite far apart from being sharp in terms of their dependence on r. Subsequently, Krivelevich [6] and separately Dudek and Prałat [5] proved that if n is sufficiently large, then $\hat{r}(P_n, r) \leq 600\, r^2 (\ln r)\, n$. As a result we get that $\hat{r}(P_n, r) = O((\ln r)r^2 n)$. In this paper, we improve the latter upper bound and prove that $\hat{r}(P_n, r) \leq (18 + o_r(1))\, r^2 (\ln r)\, n$.

Material and methods

Our method for establishing the upper bound is essentially a probabilistic construction. In this scheme, first we prove a bipartite version of Posa's Lemma which seeks for a long path in a bipartite graph. In fact, we obtain a sufficient condition for the existence of a long path in a bipartite graph. Then, using the probabilistic method and Chernoff's inequality, we prove that there exists a random sparse bipartite graph G such that every dense spanning subgraph of G contains a long path.

Results and discussion

We determine an upper bound for the multicolor size-Ramsey number of paths. This upper bound shows that $\hat{r}(P_n, r)$ grows linearly with n and its dependency on the number of colors r is of order $r^2(\ln r)$. More precisely, we prove that $\hat{r}(P_n, r) \leq 18(1 + o_r(1))r^2(\ln r)n$. This upper bound is nearly optimal, since it is well known that $\hat{r}(P_n, r) = \Omega(r^2 n)$.

Conclusion

The following conclusions were drawn from this research.

- Let $k > 0$ be an integer and $G(V_0 \cup V_1, E)$ be a bipartite graph with this property that for every subset $S \subseteq V_i$, $i \in \{0, 1\}$, with $|S| \leq k$ we have $|\Gamma(S)| \geq 2|S|$. Then, G contains a path P_{4k}.
- Let γ, f, d and ε be positive numbers with $d > \frac{1}{\gamma} \geq 1$ and $f > \frac{18(\ln d + 2)}{((1-\varepsilon)\gamma d - 1)^2}$. For sufficiently large n, there exists a bipartite graph G with nd vertices in each part, such that $e(G) \leq (1 + \varepsilon) f d^2 n$ and every spanning subgraph H of G with $e(H) \geq \Upsilon e(G)$ contains a path P_n.
- For an integer $r \geq 2$ and sufficiently large n, we have

$$\hat{r}(P_n, r) \leq 18(1 + o_r(1))r^2(\ln r)n.$$

Keywords: Random graphs; Ramsey numbers; Size-Ramsey numbers; Paths.

*Corresponding author: rjavadi@cc.iut.ac.ir