Generalization of Buchberger's Algorithm with Respect to Several Orderings on Difference Modules

Hamzeh Harfsheno, Abdolali Basiri*, Sajjad Rahmani;
School of Mathematics and Computer Science, Damaghan University, Damghan, Iran

Received: 11/23/2019 Accepted: 05/25/2019

Extended Abstract

Paper pages (513-526)

Introduction

The Gröbner basis invented by Buchberger, is an algebraic tool to solve various problems through the algorithmic methods within the polynomial ideal theory framework. Although the Gröbner basis is mainly employed in computational commutative algebra, considerable development has been recently achieved in its application in non-commutative algebra, differential algebra, and difference algebra.

Founded by Ritt and Cohn, difference algebra aims at studying algebraic difference equations in a similar way that polynomial equations are studied in commutative algebra and algebraic geometry. It has been shown that the notion of Gröbner basis is essential for many difference algebra problems.

The study of difference dimension polynomials and their computation methods is an important issue in the theory of difference algebra. Several methods have been developed to compute the dimension polynomials; these methods are mainly based either on the concepts of the characteristic set or free resolutions. The main drawback of these methods is the lack of an efficient algorithm for constructing the characteristic sets and their inability to use a free resolution on all types of systems. Through the use of a Gröbner basis with respect to several term orderings, Levin presented an algorithm for computing the difference dimension polynomial. He generalized the Gröbner basis method to the cases involving free modules over difference rings with respect to several term orderings associated with a partition of a set of variables. In this paper, a different proof is presented for Levin's result. Also, an algorithm is proposed for computing a Gröbner basis of difference modules over a ground difference field with respect to several term orderings.

Material and methods

In this scheme, let $\mathcal{D}$ be a ring of difference operators and E be a finitely generated free difference module over $\mathcal{D}$ generated by $e = \{e_1, \dots, e_m\}$. We introduce a special representation for any $f \in E$ which corresponds to the several term orderings and a subset of E. Based on such representation, we generalize the Buchberger theorem to the cases

involving free modules over difference rings with respect to several term orderings associated with a variables set partition.

Results and discussion

It was proven that every monomial ideal of a difference operator ring is finitely generated. Furthermore, it was shown that every monomial difference submodule has a finite basis. In the case of the sequel, using the notion of representation with respect to several term orderings, the necessary and sufficient condition was provided for the existence of a Gröbner basis with respect to several term orderings for a finitely-generated difference module. Finally, Levin's result is proven on the generalization of the Buchberger theorem to the case of free difference modules. In this proof, the extra conditions that were considered in Levin's proof are eliminated.

Conclusion

Gröbner basis with respect to several term orderings is a powerful tool to compute the multivariate difference dimension polynomials. In this paper, first, we proved the generalized Gröbner basis with respect to several term orderings for a free difference sub-module. Then based on these results, an algorithm was proposed to compute Gröbner basis with respect to several term orderings for a free difference sub-module. Finally, the first MAPLE implementation of the algorithm was presented.

Keywords: Difference ring, Difference module, Ring of difference operator, Gröbner bases.

*Corresponding author: basiri@du.ac.ir