Numerical Solution of Space-time Fractional two-dimensional Telegraph Equation by Shifted LEgendre Operational Matrices

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Extended Abstract

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Introduction

Fractional differential equations (FDEs) have attracted in the recent years a considerable interest due to their frequent appearance in various fields and their more accurate models of systems under consideration provided by fractional derivatives. For example, fractional derivatives have been used successfully to model frequency dependent damping behavior of many viscoelastic materials. They are also used in modeling of many chemical processed, mathematical biology and many other problems in engineering. The history and a comprehensive treatment of FDEs are provided by Podlubny and a review of some applications of FDEs are given by Mainardi.

The fractional telegraph equation has recently been considered by many authors. Cascaval et al. discussed the time-fractional telegraph equations, dealing with wellposedness and presenting a study involving asymptotic by using the Riemann-Liouville approach. Orsingher and Beghin discussed the time-fractional telegraph equation and telegraph processes with Brownian time, showing that some processes are governed by time-fractional telegraph equations. Chen et al. also discussed and derived the solution of the time-fractional telegraph equation with three kinds of nonhomogeneous boundary conditions, by the method of separating variables. Orsingher and Zhao considered the space-fractional telegraph equations, obtaining the Fourier transform of its fundamental solution and presenting a symmetric process with discontinuous trajectories, whose transition function satisfies the space-fractional telegraph equation. Momani discussed analytic and approximate solutions of the space- and time-fractional telegraph differential equations by means of the so-called Adomian decomposition method. Camargo et al. discussed the so-called general space-time fractional telegraph equations by the methods of differential and integral calculus, discussing the solution by means of the Laplace and Fourier transforms in variables t and x, respectively.

In this paper, we consider the following space-time fractional two-dimensional telegraph equation:

$$\frac{\partial^{2\alpha}u(t,x,y)}{\partial t^{2\alpha}}+p\frac{\partial^{\alpha}u\left(t,x,y\right)}{\partial t^{\alpha}}+q^2u\left(t,x,y\right)=\frac{\partial^{\beta}u(t,x,y)}{\partial x^{\beta}}+\frac{\partial^{\beta}u(t,x,y)}{\partial y^{\beta}}+f\left(t,x,y\right),$$

with initial conditions

$$u(0,x,y)=f_1(x,y), \quad u_t(0,x,y)=f_2(x,y),$$

where p, q are constants and also $1<\beta\le 2$, $0.5<\alpha\le 1$ and $x,y,t\in[0,1]$.

Material and Method

The Legendre polynomials defined on [−1, 1] are given by the following recurrence relation

$$L_{i+1}(z)=\frac{2i+1}{i+1}zL_i(z)-\frac{i}{i+1}L_{i-1}(z),\ i=1,2,...,$$

where $L_0(z)=0$,$L_1(z)=1$. The transformation $x=\frac{z+1}{2}$ transforms the interval [−1, 1] to [0, 1] and the shifted Legendre polynomials are given by

$$P_i(x)=\sum_{k=0}^{i}(-1)^{i+k}\frac{(i+k)!}{(i-k)!}\frac{x^k}{(k!)^2},i=0,1,2,3,...,$$

where $p_i(0)=(-1)^i$, $p_i(1)=1$.

In this paper, we approximate three-variable functions using the shifted Legendre polynomials. Then, we introduce the operational matrices of the Caputo fractional derivative in one-dimensional and two-dimensional cases and also the operational matrix of the Riemann-Liouville fractional integral using the shifted Legendre polynomials. By applying these concepts on the two-dimensional space-time fractional telegraph equation, the problem will be reduced to solve a system of algebraic equations that can be easily solved.

Conclusion

Three examples are presented to illustrate the efficiency, accuracy, and stability of the proposed method. The proposed numerical examples show that this method is very efficient and the obtained approximate results are very accurate in comparison with the exact solutions. Also, as expected, numerical results show that when the derivative order in the problem approaches to an integer, the obtained results also closes to the solution of the telegraph equation with the integer derivative. In addition, the numerical analysis of stability in these examples shows that the proposed method has good stability.

Keywords: Operational matrices, Caputo fractional derivative, Riemann-Liouville fractional integral, Shifted Legendre polynomials, Two-dimensional space-time fractional telegraph equation

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