Topics on the Ratliff-Rush Closure of an Ideal

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Extended Abstract

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Introduction

Let R be a Noetherian ring with unity and I be a regular ideal of R, that is, I contains a nonzerodivisor. Let $I^{n+1} :_R I^n = \{ x \in R : xI^n \subseteq I^{n+1} \}$. Then $I^n :_R I^{n-1} \subseteq I^{n+1} :_R I^n$. The union of this family, $\bigcup_{n=1}^{\infty}(I^n :_R I^{n-1})$, is an interesting ideal first studied by Ratliff and Rush in [15]. The Ratliff-Rush closure of I is defined by $\tilde{I} = \bigcup_{n=1}^{\infty}(I^n :_R I^{n-1})$. A regular ideal I for which $\tilde{I} = I$ is called Ratliff-Rush ideal.

The present paper, reviews some of the known properties, and compares properties of Ratliff-Rush closure of an ideal with its integral closure. We discuss some general properties of Ratliff-Rush ideals, consider the behaviour of the Ratliff-Rush property with respect to certain ideal and ring-theoretic operations, and try to indicate how one might determine whether a given ideal is Ratliff-Rush or not.

For a proper regular ideal I, we denote by $G(I)$ the graded ring (or form ring) $G(I) = R/I \oplus I/I^2 \oplus I^2/I^3 \oplus \cdots$. All powers of I are Ratliff-Rush ideals if and only if its positively graded ideal $G(I)_+ = I/I^2 \oplus I^2/I^3 \oplus \cdots$contains a nonzerodivisor. An ideal $J \subseteq I$ is called a reduction of I if $I^{n+1} = JI^n$ for some $n \in \mathbb{N}$. A reduction J is called a minimal reduction of I if it does not properly contain a reduction of I. The least such n is called the reduction number of I with respect to J, and denoted by $r_J(I)$. A regular ideal I is always a reduction of its associated Ratliff-Rush ideal $\tilde{I}$.

The Hilbert-Samuel function of I is the numerical function that measures the growth of the length of R/I^n for all $\in \mathbb{N}$. This function, $\lambda(R/I^n)$,is a polynomial in, for all large n. Finally, in the last section, we review some facts on Hilbert function of the Ratliff-Rush closure of an ideal. Ratliff and Rush [15, (2.4)] prove that every nonzero ideal in a Dedekind domain is concerning a Ratliff-Rush ideal. They also [15, Remark 2.5] express interest in classifying the Noetherian domains in which every nonzero ideal is a Ratliff-Rush ideal. This interest motivated the next sequence of results. A domain with this property has dimension at most one.

Results and discussion

The present paper compares properties of Ratliff-Rush closure of an ideal with its integral closure. Furthermore, ideals in which their associated graded ring has positive depth, are introduced as ideals for which all its powers are Ratliff-Rush ideals. While stating that each regular ideal is always a reduction of its associated Ratliff-Rush ideal, it expresses the command for calculating the Rutliff-Rush closure of an ideal by its reduction. This fact that Hilbert polynomial of an ideal has the same Hilbert polynomial its Ratliff-Rush closure, is from our other results.

Conclusion

The Ratliff-Rush closure of ideals is a good operation with respect to many properties, it carries information about associated primes of powers of ideals, about zerodivisors in the associated graded ring, preserves the Hilbert function of zero-dimensional ideals, etc.

Keywords: Ratliff-Rush closure, Integral closure, Hilbert polynomial, Reduction number.

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