Inert Module Extensions, Multiplicatively Closed Subsets Conserving Cyclic Submodules and Factorization in Modules

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Extended Abstract

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Introduction

Suppose that R is a commutative ring with identity, M is a unitary R-module and S is a multiplicatively closed subset of R.

Factorization theory in commutative rings, which has a long history, still gets the attention of many researchers. Although at first, the focus of this theory was factorization properties of elements in integral domains, in the late nineties the theory was generalized to commutative rings with zero-divisors and to modules. Also recently, the factorization properties of an element of a module with respect to a multiplicatively closed subset of the ring has been investigated. It has been shown that using these general views, one can derive new results and insights on the classic case of factorization theory in integral domains.

An important and attractive question in this theory is understanding how factorization properties of a ring or a module behave under localization. In particular, Anderson, et al in 1992 showed that if R is an integral domain and every principal ideal of R_S contracts to a principal ideal of R, then there are strong relations between factorization properties of R and R_S. In the same paper and also in another paper by Ağargün, et al in 2001 the concepts of inert and weakly inert extensions of rings were introduced and the relation of factorization properties of R and R_S, under the assumption that $R \subseteq R_S$ is (weakly) inert, is studied.

In this paper, we generalize the above concepts to modules and with respect to a multiplicatively closed subset. Then we utilize them to relate the factorization properties of M and M_S.

Material and methods

We first recall the concepts of factorization theory in modules with respect to a multiplicatively closed subset of the ring. Then, we define multiplicatively closed subsets conserving cyclic submodules of M and say that S conserves cyclic submodules of M, when the contraction of every cyclic submodule of M_S to M is a cyclic submodule. We present conditions on S equivalent to conserving cyclic submodules of M and study how factorization properties of M is related to those of M_S, when S coserves cyclic submodules of M. Finally we present generalizations of inert and weakly inert extensions of rings to modules and investigate how factorization properties behave under localization with respect to S, when $M \leq M_S$ is inert or weakly inert.

Results and discussion

We show that if R is an integral domain, M is torsion-free and S conserves cyclic submodules of M, then S splits M (as defined by Nikseresht in 2018) and hence factorization properties of M and those of M_S are strongly related. Also we show that under certain conditions, the converse is also true, that is, if S splits M, then S conserves cyclic submodules of M.

Suppose that T is a multiplicatively closed subset of R containing S and $T^{'} = S^{-1}T$. We show that if $M \leq M_S$ is a $(T, T^{'})$-weakly inert extension, then there is a strong relationship between T- factorization properties of M and T'-factorization properties of M_S. For example, under the above assumptions, if M is also torsion-free and has unique (or finite or bounded) factorization with respect to T, then M_S has the same property with respect to T'.

Conclusion

In this paper, the concepts of a multiplicatively closed subset conserving cyclic submodules and inert and weakly inert extensions of modules are introduced and utilized to derive relations between factorization properties of a module M and those of its localization M_S. It is seen that many properties can be delivered from one to another when S conserves cyclic submodules or when $M \leq M_S$ is a weakly inert extension, especially when R is an integral domain and M is torsion-free.

Keywords: Multiplicatively closed subsets conserving cyclic submodules, Inert extension, Atomic module, Unique factorization module.

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