A locally Convex Topology on the Beurling Algebras

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Extended Abstract
Paper pages (221-228)

Introduction
Let \( G \) be a locally compact group with a fixed left Haar measure \( \lambda \) and \( \omega \) be a weight function on \( G \); that is a Borel measurable function \( \omega : G \to (0, \infty) \) with \( \omega(xy) \leq \omega(x)\omega(y) \) for all \( x, y \in G \). We denote by \( L^1(G, \omega) \) the set of all measurable functions \( \varphi \) such that \( \varphi \omega \in L^1(G) \); the group algebra of \( G \) as defined in [2]. Then \( L^1(G, \omega) \) with the convolution product \( ** \) and the norm \( \| \cdot \|_{1,\omega} \) defined by \( \| \varphi \|_{1,\omega} = \| \varphi \omega \|_1 \) is a Banach algebra known as Beurling algebra. We denote by \( n(G, \omega) \) the topology generated by the norm \( \| \cdot \|_{1,\omega} \).

Also, let \( L^{\infty}(G, \frac{1}{\omega}) \) denote the space of all measurable functions \( f \) with \( f/\omega \in L^{\infty}(G) \), the Lebesgue space as defined in [2]. Then \( L^{\infty}(G, \frac{1}{\omega}) \) with the product \( \cdot \omega \) defined by \( f \cdot \omega g = fg/\omega \), the norm \( \| \cdot \|_{\infty,\omega} \) defined by \( \| f \|_{\infty,\omega} = \| f/\omega \|_{\infty} \), and the complex conjugation as involution is a commutative \( C^* \)-algebra. Moreover, \( L^{\infty}(G, \frac{1}{\omega}) \) is the dual of \( L^1(G, \omega) \). In fact, the mapping \( T: L^{\infty}\left(G, \frac{1}{\omega}\right) \to L^1(G, \omega)^* \), \( \langle T(f), \varphi \rangle = \int f(x)\varphi(x) \, d\lambda(x) \) is an isometric isomorphism.

We denote by \( L^{\infty}_0\left(G, \frac{1}{\omega}\right) \) the \( C^* \)-subalgebra of \( L^{\infty}(G, \frac{1}{\omega}) \) consisting of all functions \( g \) on \( G \) such that for each \( \epsilon > 0 \), there is a compact subset \( K \) of \( G \) for which \( \| gx_{G \setminus K} \|_{\infty,\omega} < \epsilon \). For a study of \( L^{\infty}_0\left(G, \frac{1}{\omega}\right) \) in the unweighted case see [3,6].

We introduce and study a locally convex topology \( \beta^1(G, \omega) \) on \( L^1(G, \omega) \) such that \( L^{\infty}_0\left(G, \frac{1}{\omega}\right) \) can be identified with the strong dual of \( L^1(G, \omega) \). Our work generalizes some interesting results of [15] for group algebras to a more general setting of weighted group algebras. We also show that \( (L^1(G, \omega), \beta^1(G, \omega)) \) could be a bornological space only if \( G \) is compact. Finally, we prove that \( L^{\infty}_0\left(G, \frac{1}{\omega}\right) \) is complemented in \( L^{\infty}(G, \frac{1}{\omega}) \) if and only if \( G \) is compact. For some similar recent studies see [4,7,8,10,12-14]. One may be interested to see the work [9] for an application of these results.

Main results
We denote by \( \mathcal{C} \) the set of increasing sequences of compact subsets of \( G \) and by \( \mathcal{R} \) the set of increasing sequences \( (r_n) \) of real numbers in \( (0, \infty) \) divergent to infinity. For any \( (C_n) \in \mathcal{C} \) and \( (r_n) \in \mathcal{R} \), set \( U((C_n), (r_n)) = \{ \varphi \in L^1(G, \omega) : \| \varphi \|_{1,\omega} < r_n, \forall n \geq 1 \} \) and note that \( U((C_n), (r_n)) \) is a convex balanced absorbing set in the space \( L^1(G, \omega) \). It is easy to see that the family \( \mathcal{U} \) of all sets \( U((C_n), (r_n)) \) is a base of neighbourhoods of zero for a locally convex topology on \( L^1(G, \omega) \); see for example [16]. We denote this topology by \( \beta^1(G, \omega) \).
Here we use some ideas from [15], where this topology has been introduced and studied for group algebras.

**Proposition 2.1** Let $G$ be a locally compact group, and $\omega$ be a weight function on $G$. The norm topology $n(G, \omega)$ on $L^1(G, \omega)$ coincides with the topology $\beta^1(G, \omega)$ if and only if $G$ is compact.

**Proposition 2.2** Let $G$ be a locally compact group, and $\omega$ be a weight function on $G$. Then the dual of $(L^1(G, \omega), \beta^1(G, \omega))$ endowed with the strong topology can be identified with $L_0^\infty \left(G, \frac{1}{\omega}\right)$ endowed with $\| \cdot \|_{\infty, \omega}$-topology.

**Proposition 2.3** Let $G$ be a locally compact group, and $\omega$ be a weight function on $G$. Then the following assertions are equivalent:

a) $(L^1(G, \omega), \beta^1(G, \omega))$ is barrelled.

b) $(L^1(G, \omega), \beta^1(G, \omega))$ is bornological.

c) $(L^1(G, \omega), \beta^1(G, \omega))$ is metrizable.

d) $G$ is compact.

**Proposition 2.4** Let $G$ be a locally compact group, and $\omega$ be a weight function on $G$. Then $L_0^\infty \left(G, \frac{1}{\omega}\right)$ is not complemented in $L^\infty(G, \frac{1}{\omega})$.

**Keywords:** Locally compact group, Locally convex topology, Weighted Lebesgue space, Dual.

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