# A locally Convex Topology on the Beurling Algebras

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**Extended Abstract**

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## Introduction

Let $G$ be a locally compact group with a fixed left Haar measure $\lambda$ and $\omega$ be a weight function on $G$; that is a Borel measurable function $\omega: G \longrightarrow (0,\infty)$ with $\omega(xy) \leq \omega(x)\omega(y)$ for all $x, y \in G$. We denote by $L^1(G,\omega)$ the set of all measurable functions $\varphi$ such that $\varphi\omega \in L^1(G)$; the group algebra of $G$ as defined in [2]. Then $L^1(G,\omega)$ with the convolution product "*" and the norm $\|.\|_{1,\omega}$ defined by $\|\varphi\|_{1,\omega} = \|\varphi\omega\|_1$ is a Banach algebra known as Beurling algebra. We denote by $n(G,\omega)$ the topology generated by the norm $\|.\|_{1,\omega}$. Also, let $L^\infty(G,\frac{1}{\omega})$ denote the space of all measurable functions $f$ with $f/\omega \in L^\infty(G)$, the Lebesgue space as defined in [2].

Then $L^\infty(G,\frac{1}{\omega})$ with the product $\cdot_\omega$ defined by $f\cdot_\omega \mathrm{g} = f\mathrm{g}/\omega$, the norm $\|.\|_{\infty,\omega}$ defined by $\|f\|_{\infty,\omega} = \|f\omega\|_\infty$ , and the complex conjugation as involution is a commutative $\mathrm{C}^*$-algebra. Moreover, $L^\infty(G,\frac{1}{\omega})$ is the dual of $L^1(G,\omega)$. In fact, the mapping $T: L^\infty\left(G,\frac{1}{\omega}\right) \longrightarrow L^1(G,\omega)^*$ , $< T(f),\varphi > = \int f(x)\varphi(x)\, d\lambda(x)$ is an isometric isomorphism.

We denote by $L_0^\infty\left(G,\frac{1}{\omega}\right)$ the $\mathrm{C}^*$-subalgebra of $L^\infty(G,\frac{1}{\omega})$ consisting of all functions $g$ on $G$ such that for each $\epsilon > 0$, there is a compact subset $K$ of $G$ for which $\|\mathrm{g}\chi_{G\setminus K}\|_{\infty,\omega} < \epsilon$. For a study of $L_0^\infty\left(G,\frac{1}{\omega}\right)$ in the unweighted case see [3,6].

We introduce and study a locally convex topology $\beta^1(G,\omega)$ on $L^1(G,\omega)$ such that $L_0^\infty\left(G,\frac{1}{\omega}\right)$ can be identified with the strong dual of $L^1(G,\omega)$. Our work generalizes some interesting results of [15] for group algebras to a more general setting of weighted group algebras. We also show that $(L^1(G,\omega),\beta^1(G,\omega))$ could be a normable or bornological space only if $G$ is compact. Finally, we prove that $L_0^\infty\left(G,\frac{1}{\omega}\right)$ is complemented in $L^\infty(G,\frac{1}{\omega})$ if and only if G is compact. For some similar recent studies see [4,7,8,10,12-14]. One may be interested to see the work [9] for an application of these results.

## Main results

We denote by $\mathcal{C}$ the set of increasing sequences of compact subsets of G and by $\mathcal{R}$ the set of increasing sequences $(\mathrm{r_n})$ of real numbers in $(0,\infty)$ divergent to infinity. For any $(\mathrm{C_n}) \in \mathcal{C}$ and $(\mathrm{r_n}) \in \mathcal{R}$, set $\mathrm{U}((\mathrm{C_n}),(\mathrm{r_n})) = \{\varphi \in L^1(G,\omega): \|\varphi\|_{1,\omega} < \mathrm{r_n}, \forall \mathrm{n} \geq 1\}$ and note that $\mathrm{U}((\mathrm{C_n}),(\mathrm{r_n}))$ is a convex balanced absorbing set in the space $L^1(G,\omega)$. It is easy to see that the family $\mathcal{U}$ of all sets $\mathrm{U}((\mathrm{C_n}),(\mathrm{r_n}))$ is a base of neighbourhoods of zero for a locally convex topology on $L^1(G,\omega)$; see for example [16]. We denote this topology by $\beta^1(G,\omega)$.

Here we use some ideas from [15], where this topology has been introduced and studied for group algebras.

**Proposition 2.1** Let $G$ be a locally compact group, and $\omega$ be a weight function on $G$. The norm topology $n(G,\omega)$ on $L^1(G,\omega)$ coincides with the topology $\beta^1(G,\omega)$ if and only if $G$ is compact.

**Proposition 2.2** Let $G$ be a locally compact group, and $\omega$ be a weight function on $G$. Then the dual of $(L^1(G,\omega), \beta^1(G,\omega))$ endowed with the strong topology can be identified with $L_0^\infty\left(G, \frac{1}{\omega}\right)$endowed with $\|.\|_{\infty,\omega}$-topology.

**Proposition 2.3** Let $G$ be a locally compact group, and $\omega$ be a weight function on $G$. Then the following assertions are equivalent:

a) $(L^1(G,\omega), \beta^1(G,\omega))$ is barrelled.

b) $(L^1(G,\omega), \beta^1(G,\omega))$ is bornological.

c) $(L^1(G,\omega), \beta^1(G,\omega))$ is metrizable.

d) $G$ is compact.

**Proposition 2.4** Let $G$ be a locally compact group, and $\omega$ be a weight function on $G$. Then $L_0^\infty\left(G, \frac{1}{\omega}\right)$ is not complemented in $L^\infty(G, \frac{1}{\omega})$.

**Keywords**: Locally compact group, Locally convex topology, Weighted Lebesgue space, Dual.

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