Generalized Iterative Methods for Solving Double Saddle Point Problem

Michele Benzi[1], Fatemeh Panjeh Ali Beik[*2],
Seyyed Hassan Azizi Chaparpordi[2], Zohreh Roygar[2]
1. Scuola Normale Superiore, Piazza dei Cavalieri 7, 56126 Pisa, Italy
2. Department of Mathematics, Vali-e-Asr University of Rafsanjan, Rafsanjan, Iran

Received: 6 Nov. 2017 Accepted: 8 August 2018

Extended Abstract

Paper pages (137-150)

Introduction

Consider the following linear system of equations:

$$\mathcal{A}u \equiv \begin{bmatrix} A & B^T & C^T \\ B & 0 & 0 \\ C & 0 & -D \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \equiv b, \tag{1}$$

where $A \in \mathbb{R}^{n\times n}$ and $D \in \mathbb{R}^{p\times p}$ are symmetric positive definite matrices (SPD), $B \in \mathbb{R}^{m\times n}$ and $C \in \mathbb{R}^{p\times n}$ $(m \le n)$ such that B has full column rank. Linear systems of this type arise, e.g., from finite element models of liquid crystals; see [1,5] for further details. In [1], it is shown if A and D are SPD, then $\mathcal{A}$ is nonsingular if and only if B is a matrix with full column rank.

The decomposition $\mathcal{A} = \mathcal{M} - \mathcal{N}$ is called splitting if $\mathcal{M}$ is nonsingular. Basically, a stationary iterative method to solve $\mathcal{A}u = b$ is obtained by using the splitting $\mathcal{A} = \mathcal{M} - \mathcal{N}$ as follows:

$$u_{k+1} = \mathcal{G}u_k + \mathcal{M}^{-1}b, \qquad k = 0,1,2,\ldots, \tag{2}$$

where $\mathcal{G} = \mathcal{M}^{-1}\mathcal{N}$ is called iteration matrix and the initial guess u_0 is given. The above iterative method converges to the exact solution of (1) for any choice of the initial guess u_0 if and only if the spectral radius of its iteration matrix is strictly less than one, i.e, $\rho(\mathcal{G}) < 1$.

Evidently, one may consider (1) as a stabilized saddle point problem in the following from:

$$\begin{bmatrix} A_{11} & C^T \\ C & -D \end{bmatrix} u = b,$$

where,

$$A_{11} = \begin{bmatrix} A & B^T \\ B & 0 \end{bmatrix}.$$

As seen, the matrix A_{11} has a similar structure to the coefficient matrix associated with a saddle point problem. Over the years, different types of iterative schemes have been proposed in the literature for solving saddle point problems, we refer the reader to the survey [3] for developments up to about 2005. Recently, for stabilized saddle point problems, some new

results have been established related to block diagonal and triangular splittings and their corresponding iterative schemes and extracted preconditioners, see [2] for more details.

The main purpose of this paper is to develop some iterative methods for solving double saddle point problem (1). To this end, we first consider the following decomposition for the coefficient matrix:

$$\mathcal{A} = \mathcal{D} - \mathcal{L} - \mathcal{U}, \tag{3}$$

where

$$\mathcal{D} = \begin{bmatrix} A & B^T & 0 \\ B & 0 & 0 \\ 0 & 0 & -D \end{bmatrix}, \quad \mathcal{L} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -C & 0 & 0 \end{bmatrix}, \quad \mathcal{U} = \begin{bmatrix} 0 & 0 & -C^T \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Then, block iterative methods are constructed by using some block splittings which are natural extensions of some well-known splittings in the literature.

Material and methods

We mainly develop some stationary iterative schemes in the form (2) for solving double saddle point problem using the block decomposition (3). First, we present the block Jacobi iterative method and study its convergence properties under certain condition. Then, using a relaxation parameter, the weighted version of the block Jacobi method together with its convergence analysis are considered. Moreover, we extend a method from the class of block Gauss-Seidel (GS) iterative method and establish its convergence properties. Furthermore, the block successive overrelaxation (SOR) splitting is used to construct an iterative scheme to solve the mentioned double saddle point problem and its convergence properties are analyzed. In order to illustrate the efficiency of the proposed methods, we report some numerical experiments for a class of double saddle point problems arising from the modeling of liquid crystal directors using finite elements.

Results and discussion

Several theorems are established to prove that spectral radii of iteration matrices corresponding to each of proposed iterative schemes are strictly less than one. Iterative schemes of block GS-type and block SOR-type rely on two fixed parameters, two formulas are suggested for choosing appropriate values for each of these parameters. It is well-known that for a given splitting $\mathcal{A} = \mathcal{M} - \mathcal{N}$, the matrix $\mathcal{M}^{-1}$ may be exploited as a preconditioner in conjunction with Krylov subspace methods to accelerate their convergence speeds. The examined numerical results for stationary iterative methods suggested that extracted preconditioners from these splittings might have optimal properties. Therefore, the performances of these precnditioners in conjunction with flexible generalized minimum residual (FGMRES) method are also reported which confirm the good performance of preconditioners.

Conclusion

We proved some convergence results for block iterative schemes to solve double saddle point problems. The mentioned iterative methods were obtained by using some block splittings. The presented results can be also used to study eigenvalue distributions of preconditioned matrices corresponding to mentioned splittings. Part of results are proved under the condition

$A \succ C^T D^{-1} C$ that holds when (1) arises from finite element modeling of liquid crystal directors. For a given $r > 0$, evidently, the following linear system is equivalent to (1),

$$\begin{bmatrix} \tilde{A} & B^T & C^T \\ B & 0 & 0 \\ C & 0 & -D \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \tilde{b}_1 \\ b_2 \\ b_3 \end{bmatrix}, \tag{4}$$

where $\tilde{A} = A + rB^T B$ and $\tilde{b}_1 = b_1 + rB^T b_2$. For the above linear system of equations, there are some cases that the condition $\tilde{A} \succ C^T D^{-1} C$ can be satisfied by choosing large values for r. Choosing suitable values for r and efficient preconditioners to directly solve (4) left as a subject for future researches. The idea of solving (4) instead of (1) is taken from the idea used in [4].

References

1. Beik F. P. A., Benzi M., "Iterative Methods for Double Saddle Point Systems", SIAM J. Matrix Anal. Appl. 39 (2018) 902-921.
2. Beik F. P.A, Benzi M., Chaparpordi S. H. A., "On block diagonal and block triangular iterative schemes and preconditioners for stabilized saddle point problems," J. Comput. Appl. Math. 326 (2017) 15-30.
3. Benzi M., Golub G. H., Liesen J., "Numerical solution of saddle point problems", Acta Numer. 14 (2005) 1-137.
4. Benzi M., Wang Z., "Analysis of augmented Lagrangian-based preconditioners for the steady incompressible Navier-Stokes equations," SIAM J. Sci. Comput. 33 (2011) 2761-2784.
5. Ramage A., Gartland Jr E. C., "A preconditioned nullspace method for liquid crystal director modeling," SIAM J. Sci. Comput. 35 (2013) B226-B247.

Keywords: Double saddle point problem, Block Gauss-Seidel method, Block SOR method, Convergence, Liquid crystal..

*Corresponding author: f.beik@vru.ac.ir