Some Properties of Arveson Spectrum on Locally Compact Hypergroups

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Extended Abstract

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Introduction

Arveson and Connes spectrums are key tools for classification of Von Neumann algebras. A.R. Medghalchi and S.M. Tabatabaie initiated Arveson spectrum and spectral subspaces on locally compact hypergroup. In this paper we extend some basic properties of Arveson spectrum on some important classes of hypergroups which were introduced by Dunkl and Ramirez. Many significant results about Connes spectrum come from this property. Actually, we show that $\mathrm{sp}_\pi(x_1 x_2) \subseteq \mathrm{sp}_\pi(x_1) * \mathrm{sp}_\pi(x_2)$, where $x_1, x_2 \in M$ and (M, K, π) is a W^*–system. A C^*–algebra M is called W^*–algebra if for a Banach algebra M_*, $(M_*)^* = M$, where $(M_*)^*$ is the conjugate space of M_*. We denote by $B_\sigma(M)$ the set of all $(\sigma(M, M_*), \sigma(M, M_*))$–continuous operators on a W^*–algebra M. For a locally compact Hausdorff space K, we will denote by $M(K)$ the set of all complex Radon measures on K, and by $M^+(K)$ the set of all non-negative elements of $M(K)$. The support of any $\mu \in M(K)$ will be denoted by $\mathrm{supp}\,\mu$. Also, we will denote by δ_x the Dirac measure at the point *x*.

Let K be a non-empty locally compact Hausdorff space with the following properties:

(1) there exists a binary operation $*$ on $M(K)$ such that $M(K)$ with this operation is a complex associative algebra;

(2) for all $\mu, \nu \in M^+(K)$, $\mu * \nu \in M^+(K)$ and the mapping $(\mu, \nu) \mapsto \mu * \nu$ from $M^+(K) \times M^+(K)$ to $M^+(K)$ is continuous, where $M^+(K)$ is equipped with cone topology;

(3) for all $x, y \in K$, $\delta_x * \delta_y$ is a probability measure with compact support;

(4) the mapping $(x, y) \mapsto \mathrm{supp}(\delta_x * \delta_y)$ from $K \times K$ into the space $\varphi(K)$ of all non-empty compact subsets of K is continuous, where $\varphi(K)$ is equipped with Michael topology;

(5) there exists an element $e \in K$ such that for each $x \in K$, $\delta_x * \delta_e = \delta_e * \delta_x = \delta_x$;

(6) there exists a homeomorphism $x \mapsto x^-$ from K onto K such that for all $x, y \in K$, $(x^-)^- = x$ and $(\delta_x * \delta_y)^- = \delta_{y^-} * \delta_{x^-}$;

(7) for all $x, y \in K$, $e \in \mathrm{supp}(\delta_x * \delta_y)$ if and only if $x = y^-$;

Then $K \equiv (K, *, ^-, e)$ is called a hypergroup.

Main Results

If A is a commutative Banach algebra and $E \subseteq A$, the hull of E is defined by
$\text{hull}(E) := \{\varphi \in \Delta(A) : \ \forall a \in E \ , \ \hat{a}(\varphi) = 0\}$, where $\Delta(A)$ is the structure space of A.

Definition 1. Let M be a W^*-algebra and K be a commutative locally compact hypergroup. A norm-decreasing algebra-homomorphism $\pi : M(K) \longrightarrow B_\sigma(M)$ is called a representation if

(1) for each $t \in K$, $\pi_t : M \to M$ is an *-automorphism;

(2) for each $x \in M$ and $p \in M_*$, the function $t \mapsto \langle \pi_t(x), p \rangle$ is continuous;

(3) $\pi_e = I_M$, where e is the identity of K and I_M is the identity mapping on M.

In this case (M, K, π) is called a W^*-system.

Throughout this paper, M is a W^*-algebra and π is a representation of hypergroup K on M.

Definition 2.

(1) The Arveson spectrum of π is defined by $\text{sp}\,\pi := \text{hull}(\{ f \in L^1(K) : \pi(f) = 0\})$.

(2) For each $x \in M$, we put $\text{sp}_\pi(x) := \text{hull}(\{ f \in L^1(K) : \pi(f)(x) = 0\})$.

(3) Let E be a closed subset of $\hat{K}$. We define the associated spectral subspace by
$M(\pi, E) := \{x \in M : \text{sp}_\pi(x) \subseteq E\}$.

Remark 3.We say that a hypergroup K satisfies condition (α) if for any two compact subsets E_1 and

E_2 of $\hat{K}$ and every f in $L^1(K)$ with $\hat{f} \equiv 0$ on a neighborhood of $E_1 * E_2$, there exist $f_1, f_2 \in L^1(K)$ such that $\hat{f}_i \equiv 1$ on a neighborhood of E_i ($i = 1, 2$), and for any $s_1, s_2 \in K$ and almost any $t \in \text{supp}(\delta_{s_1} * \delta_{s_2})$,

$$\kappa(s_1, s_2, t) := \int f(s)\, f_1(s_1 * s^-)\, f_2(t * s^-)\, dm(s) = 0.$$

We show that locally compact groups and also some important classes of hypergroups, introduced by Dunkl and Ramirez, satisfy the condition (α).

Theorem 4. Let (M, K, π) be a W^*–system and K satisfy in the condition (α). Then for all $x_1, x_2 \in M$, $\text{sp}_\pi(x_1 x_2) \subseteq \text{sp}_\pi(x_1) * \text{sp}_\pi(x_2)$.

Theorem 5. Let (M, K, π) be a W^*-system, K satisfy in the condition (α), E_1 and E_2 are closed subsets of $\hat{K}$, and $E = \overline{E_1 * E_2}$. Then for each $x_1 \in M(\pi, E_1)$ and $x_2 \in M(\pi, E_2)$ we have $x_1 x_2 \in M(\pi, E)$. On the other words, $M(\pi, E_1) M(\pi, E_2) \subseteq M(\pi, \overline{E_1 * E_2})$.

Keywords: Hypergroup, Arveson spectrum, Spectral subspace, W^*– algebra.

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