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On the Altermatic Number of Graphs

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Extended Abstract

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In 1955, Kneser conjectured that the chromatic number of the Kneser graph $\mathrm{KG}(n,k)$ is equal to $n-2k+2$. Kneser's conjecture was proved by Lovász using the Borsuk-Ulam theorem; all subsequent proofs, extensions and generalizations also relied on Algebraic Topology results, namely the Borsuk-Ulam theorem and its extensions. In 2000, Matoušek provided a combinatorial proof of Kneser's conjecture. In his proof, he used only an entirely combinatorial special case of the Tucker lemma which can be proved purely combinatorial as well.

A *hypergraph* $\mathcal{H}$ is an ordered pair $(V(\mathcal{H}),E(\mathcal{H}))$, where $V(\mathcal{H})$ is a nonempty finite set and $E(\mathcal{H})$ is a family of distinct nonempty subsets of $V(\mathcal{H})$. A *proper* t*-coloring* of a hypergraph $\mathcal{H}$ is a mapping $c:V(\mathcal{H})\longrightarrow[t]$ such that no hyperedge is monochromatic, i.e., $|c(e)|>1$ for any hyperedge $e\in E(\mathcal{H})$. The minimum integer t such that $\mathcal{H}$ admits a t-coloring is called the *chromatic number of* $\mathcal{H}$ and is denoted by $\chi(\mathcal{H})$. If a hypergraph contains some hyperedge of cardinality 1, we define its chromatic number to be infinite.

For a hypergraph $\mathcal{H}=(V(\mathcal{H}),E(\mathcal{H}))$, the *general Kneser graph* $\mathrm{KG}(\mathcal{H})$ has all hyperedges of $\mathcal{H}$ as vertex set and two vertices are adjacent if their corresponding hyperedges are disjoint. The hypergraph $\mathcal{H}$ provides a *Kneser representation* for a graph G whenever G and $\mathrm{KG}(\mathcal{H})$ are isomorphic. It is simple to see that a graph G has infinitely many Kneser representations.

Generalizing the result by Lovász, Dol'nikov (1988) proved that the colorability defect of a hypergraph $\mathcal{H}$, a combinatorial invariant of $\mathcal{H}$, provides a lower bound for the chromatic number of $\mathrm{KG}(\mathcal{H})$. Although colorability defect is a combinatorial parameter, Dol'nikov's proof still relied on topological tools. Alishahi and Hajiabolhassan, strengthening Dol'nikov's result, introduced a new combinatorial parameter as an invariant of the hypergraph at hand $\mathcal{H}$, called the *alternation number of* $\mathcal{H}$, which provides a significantly better lower bound for the chromatic number of $\mathrm{KG}(\mathcal{H})$ rather than the colorability defect of $\mathcal{H}$. Answering to some open problems, they also computed the chromatic number of some families of graphs. To state their result, we first need to set up some notations.

For a vector $X=(x_1,\dots,x_n)\in\{R,0,B\}^n$, a subsequence $x_{i_1},\dots,x_{i_t}$ of nonzero terms of X, where $1\le i_1<\cdots<i_t\le n$, is called an *alternating subsequence of* X if any two consecutive terms in this subsequence are different. We denote by $alt(X)$ the length of a longest alternating subsequence of X. Moreover, we define $alt(0,\dots,0)=0$.

Let $L_{V(\mathcal{H})}=\{v_{i_1}<\cdots<v_{i_n}:(i_1,\dots,i_n)\in S_n\}$ be the set of all linear orderings of $V(\mathcal{H})=\{v_1,\dots,v_n\}$. For any $X=(x_1,\dots,x_n)\in\{R,0,B\}^n$ and any linear ordering $\sigma: v_{i_1}<\cdots<v_{i_n}\in L_{V(\mathcal{H})}$, define $X_\sigma^R=\{v_{i_j}:x_j=R\}$, $X_\sigma^B=\{v_{i_k}:x_k=B\}$, and $X_\sigma=(X_\sigma^R,X_\sigma^B)$. Also, set $\mathcal{H}_{|X_\sigma}$ to be the hypergraph with vertex set $X_\sigma^R\cup X_\sigma^B$ and edge set

$$E(\mathcal{H}_{|X_\sigma})=\left\{A\in E(\mathcal{H}):\ A\subseteq X_\sigma^R \text{ or } A\subseteq X_\sigma^B\right\}.$$

For any $\sigma\in L_{V(\mathcal{H})}$ and positive integer k, define $alt_\sigma(\mathcal{H},k)$ to be the largest integer t such that there exists an $X\in\{R,0,B\}^n$ with $alt(X)=t$ and the chromatic number of $\mathrm{KG}(\mathcal{H}_{|X_\sigma})$ is at most . Note that when $k=1$, it means that $\mathcal{H}_{|X_\sigma}$ contains no hyperedge of $\mathcal{H}$. Now define

$$alt(\mathcal{H},k)=\min\left\{alt_\sigma(\mathcal{H},k):\sigma\in L_{V(\mathcal{H})}\right\}.$$

Let G be a graph and k be a positive integer such that $1\le k\le\chi(G)+1$. The k^{th} *altermatic number of* G, denoted by $\zeta(G,k)$, is defined as follows:

$$\zeta(G,k)=\max_{\mathcal{H}}\{|V(\mathcal{H})|-alt(\mathcal{H},k)+k-1:\mathrm{KG}(\mathcal{H})\longleftrightarrow G\},$$

where $\mathrm{KG}(\mathcal{H})\longleftrightarrow G$ means there are some homomorphisms from G to H and also from H to G. One can see that if $k=\chi(G)+1$, then for any hypergraph $\mathcal{H}$ such that $\mathrm{KG}(\mathcal{H})\longleftrightarrow G$, we have $k=\chi(\mathrm{KG}(\mathcal{H}))+1$ and $alt(\mathcal{H},k)=|V|$; and consequently, $\chi(G)=\zeta(G,k)$.

Theorem 1.(Alishahi and Hajiabolhassan 2015) *For any graph G and positive integer k, where $k\le\chi(G)+1$, we have $\chi(G)\ge\zeta(G,k)$.*

In the proof of Theorem 1, Alishahi and Hajiabolhassan crucially used Tucker's lemma, a combinatorial counterpart of the Borsuk-Ulam theorem, which is a well-known and applicable result in topological combinatorics. Evidently, it would be very nice if we have a combinatorial proof for Theorem 1, not only for its own right but also since it might inspire further study on applications of topology to problems in combinatorics. In this work, inspired by an elegant proof of Matoušek (2004), we present a purely combinatorial proof for this theorem.

Keywords: Kneser graphs; Chromatic number of graphs; Alternation number of graphs.

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