A Review on Classes of Composition Operators

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Extended Abstract

Paper pages (243-260)

Introduction

In 1976, A. Lambert characterized subnormal weighted shifts. Then he studied hyponormal weighted composition operators on $L^2(\Sigma)$ in 1986 and in 1988 subnormal composition operators studied again by him. Recently, A. Lambert, et al., have published an interesting paper: Separation partial normality classes with composition operators (2005). In 1978, R. Whitley showed that a composition operator C_T is normal if and only if $T^{-1}\Sigma = \Sigma$ essentially. Normal and quasinormal weighted composition operators were worked by J.T. Campbell, et al. in 1991. In 1993, J.T. Campbell, et al. worked also seminormal composition operators. Burnap C. and Jung I.B. studied composition operators with weak hyponormality in 2008.

Material and methods

Let $(X.\Sigma.\mu)$ be a complete σ -finite measure space and $(X.\mathcal{A}.\mu)$ be a complete σ -finite measure space where $\mathcal{A}$ is a subalgebra of Σ. For any non-negative Σ-measurable functions f as well as for any $f \in L^p(\Sigma)$, by the Radon-Nikodym theorem, there exists a unique $\mathcal{A}$-measurable function $E^{\mathcal{A}}(f)$ such that $\int_A E^{\mathcal{A}}(f)d\mu = \int_A f d\mu$ for all $A \in \mathcal{A}$. As an operator, $E^{\mathcal{A}}: L^2(\Sigma) \to L^2(\mathcal{A})$ is a contractive orthogonal projection which is called the *conditional expectation operator* with respect $\mathcal{A}$.

For a non-singular transformation $T: X \to X$, again by the Radon-Nikodym theorem, there exists a non-negative unique function $h \in L^1(\Sigma)$ such that $\mu \circ T^{-1}(S) = \int_S h d\mu,\ S \in \Sigma$. The function $h := \frac{d\mu \circ T^{-1}}{d\mu}$ is called *Radon-Nikodym derivative* of $\mu \circ T^{-1}$ with respect μ. These are two most useful tools which play important roles in this review.

For a non-negative finite-valued Σ - measurable function u and a non-singular transformation $T: X \to X$, the *weighted composition operator* W on $L^2(\Sigma)$ induced by T and u, is given by $Wf = (uC_T)f = uf o T$,
where C_T is called the *composition operator* on $L^2(\Sigma)$. W is bounded on $L^p(\Sigma)$ for $1 \le p < \infty$ if and only if $J := hE^{\mathcal{A}}(|u|^p)oT^{-1} \in L^\infty(\Sigma)$.

Results and discussion

In this paper, we review some known classes of composition operators, weighted composition operators, their adjoints and Aluthge transformations on $L^2(\Sigma)$ such as normal, subnormal, normaloid, hyponormal, p-hyponormal, p-quasihyponormal, p-paranormal, and weakly hyponormal, Furthermore, miscellaneous examples are given to illustrate that weighted composition operators lie between these classes. We discuss from the point of view of measure theory and all results depend strongly to the Radon-Nikodym derivative h and the conditional expectation operator $E^{\mathcal{A}}$ with their various types. Hence we study their fundamental properties in sections 1 and 2. Then, we review some results by A. Lambert, D.J. Harringston, R. Whitley, J.T. Campbell and W.E. Hornor.

Conclusion

According to the given miscellaneous examples in the final section, we can conclude that composition and weighted composition operators lie between these classes.

Keywords: Composition operators, Conditional expectation, Normal; Subnormal; Hyponormal, Weakly hyponormal.

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