Low Dimensional Flat Manifolds with Some Elasses of Finsler Metric

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Extended Abstract

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Introduction

An n-dimensional Riemannian manifold $(M, \alpha = \sqrt{a_{ij}(x)y^i y^j}$ is said to be *locally flat* (or locally Euclidean) if (M, α) locally isometric with the Euclidean space, that is, M admits a covering of coordinates neighborhoods each of which is isometric with a Euclidean domain. A Riemannian manifold (M, α) is locally flat if and only if M admits a covering of coordinates neighborhoods on each of, the function $\alpha(x, y)$ is independent of x. A classical result affirms that a Riemannian manifold is flat if and only if its Riemann curvature vanishes (equivalently, the sectional curvatureK_α; This is usually taken as the definition of a locally flat Riemannian manifold in the contexts. The universal Riemannian covering space of a complete and locally flat Riemannian manifold is the Euclidean space $\mathbb{E}^n = (\mathbb{R}^n, \alpha_0 = \sqrt{\delta_{ij}(x)y^i y^j}$. Up to local isometry, Bieberbach proved that any compact locally flat Riemannian manifold, is realized as a quotient space $\frac{\mathbb{R}^n}{\Gamma}$, where Γ is a discrete, co-compact and torsion free subgroup of the Euclidean group $E(n) = Isom(\mathbb{E}^n) = O(n) \ltimes \mathbb{R}^n$, cf. [2]. The only 1 dimensional complete, locally flat and connected manifolds are $\mathbb{R}$ and $\mathbb{S}^1$. In 2 dimensions, the only complete, locally flat and connected manifolds are cylinder, Möbius strip, Torus and Klein bottle. In 3 dimensions, there are only 10 complete, locally flat and connected manifolds including 6 oriented and 4 non-oriented manifolds, cf. [7].

Likewise the Riemannian case, a Finslerian manifold $(M, F = \sqrt{g_{ij}(x, y)y^i y^j})$ is said to be *locally flat* (or locally Minkowskian) if, M admits a covering of coordinates neighborhoods each of which isometric with a single Minkowski normed domain. A Finslerian manifold (M, F) is locally flat if and only if it admits a covering of coordinates neighborhoods on each of, the function $F(x, y)$ is independent of x. The flag curvature $\boldsymbol{K}$ of any locally flat Finsler manifold vanishes identically.

Material and methods

Thanks to the works of Bieberbach and Schoenflies, we apply an group theoretic approach to classify locally flat Randers manifolds. The key idea is that the isometry group of Randers manifold is a subgroup of the Euclidean group. This fact, may ease our approach to find and count discrete, co-compact and torsion frees subgroups of $\mathbb{E}(n)$. First we find the Bieberbach subgroups and then, we count those that could form an isometry subgroup.

Results and discussion

Here, flatness of a generic Finsler manifold is aimed to be defined so that it generalizes the flatness for Riemannian manifolds. The following result outcome in dimension 2 and 3:

Theorem 1. The only connected and closed n-dimensional ($n = 2,3$) closed locally flat Randers manifolds is the torus $\mathbb{T}^2$ and $\mathbb{T}^3$, respectively.

To classify the locally flat Randers manifolds, we find out that the locally flat Randers manifolds are locally flat Riemannian manifolds. Besides, the isometry group of a Randers manifold $(M, F = \alpha + \beta)$ is a subgroup of the isometry group $Isom(M, \alpha)$. Our discussion also apply the following results:

- Every dimensional locally flat Randers manifold is itself a locally flat Riemannian manifolds.
- Every dimensional locally flat Randers manifold is orientable.
- The non-Riemannian properties for generic Finsler metrics may cause obstructions for a Finsler manifold to be locally falt.

Conclusion

The following conclusions were drawn from this research.

- In dimensions 2 and 3, the only connected and closed locally flat Randers manifolds are the tori $\mathbb{T}^2$ and $\mathbb{T}^3$, respectively.
- Every dimensional locally flat Randers manifold is itself a locally flat Riemannian manifolds.

Keywords: 4th-root metric, flat manifold, isometry, Bieberbach group, Randers metric.

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