On Conformal Trans Formation of Special Curvature of Kropina Metrics

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Extended Abstract

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Introduction

Let F be a Finsler metric on an n-dimensional manifold M. For a non-zero vector $y \in T_xM$, F induces an inner product g_y on T_xM by

$$g_y(u.v) \coloneqq g_{ij}(x.y)u^iv^j = \frac{1}{2}[F^2]_{y^iy^j}u^iv^j.$$

For two arbitrary non-zero vectors v , y on T_xM the angle $\theta(y.v)$ between y and v is denoted by

$$\cos\theta\,(y.v) \coloneqq y_iv^i/F(x.y)\sqrt{g_{ij}(x.y)v^iv^j}.$$

where $y_i = g_{ij}(x.y)y^j$. It should be remark that the notion of angle is not symmetric, that is the angle $\theta(v.y)$ between y and v is different from the angle $\theta(y.v)$ between v and y generally. Now assume that F and $\bar{F}$ are two Finsler metrics on an n-dimensional manifold M. If the angle $\theta(v.y)$ with respect to F is equal to the angle $\bar{\theta}(v.y)$ with respect to $\bar{F}$ for any vectors $v.y \in T_xM\backslash\{0\}$ and any $x \in M$ then F is called conformal to $\bar{F}$ and the transformation $F \to \bar{F}$ of the metric is called conformal transformation.

The study of conformal geometry has a long and venerable history. From the beginning, conformal geometry has played an important role in physical theories then the conformal properties of a Finsler metric deserve extra attention. In conformal geometry, it is one of interesting issues to study the conformal transformation of different curvatures. In 2007, S. Basco and X. Cheng obtained the relations between some geometric quantities of two conformally related Finsler metrics and discussed the properties of those conformal transformations which preserve these quantities. Later G. Chen, Q. He and Z. Shen proved that if both conformally related $(\alpha.\beta)$-metrics F and $\bar{F}$ are Douglas metrics of non-Randers type, then the conformal transformation must be homothety and also conformal transformation between two Finsler metrics of isotropic S-curvature must be homothety.

In Finsler geometry there is an important class of metrics called $(\alpha.\beta)$-metric. An $(\alpha.\beta)$-metric is a scalar function F on TM denoted by $F \coloneqq \alpha\varphi(s)$ where $s = \frac{\beta}{\alpha}$ and $\varphi(s)$ is a C^∞ function on $(-b_0.b_0)$ with certain regularity, $\alpha \coloneqq \alpha(x.y) = \sqrt{a_{ij}(x)y^iy^j}$ is a Riemannian metric and $\beta \coloneqq \beta(x.y) = b_i(x)y^i$is a 1-form on M. Kropina metric which has the form $F \coloneqq \frac{\alpha^2}{\beta}$ is $(\alpha.\beta)$-metric with $\varphi(s) = \frac{1}{s}$ and has many applications in physic, magnetic field and dynamic systems. We are interested to study the conformal transformations between two Kropina metrics.

In this paper, conformal transformations of χ-curvature and H-curvature of Kropina metrics are studied and by using irrationality of α the conditions that preserve this quantities are investigated. Then we try to find some special cases that they lead to the conformal transformation reduces to homothetic transformation.

Conclusion

The following conclusions were drawn from this research.

Theorem: Let $F$and $\bar{F}$ be two conformally related Kropina metrics i.e. $\bar{F} = \sigma(x)F$. Then $\chi = \bar{\chi}$ if there is a scalar function $c = c(x)$ on M such that

$$\beta\sigma_0 - r_{00} = c(x)\alpha^2.$$

Remark: In special case let $r_{00} = \lambda(x)\alpha^2$, then χ-curvature is invariant under conformal transformation if the conformal transformation is a homothety.

Theorem: Let F and $\bar{F}$ be two conformally related Kropina metrics on M i.e. $\bar{F} = \sigma(x)F$.Then $H = \bar{H}$ if there is a scalar function $c = c(x)$ on M such that

$$\beta\sigma_0 - r_{00} = c(x)\alpha^2.$$

Remark: In special case let $r_{00} = \lambda(x)\alpha^2$, then H-curvature is invariant under conformal transformation if the conformal transformation is a homothety.

Keywords: Conformal transformation, Finsler metric, Kropina metrics, χ-curvature, H-curvature.

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