Inverse Spectral Problems for Sturm-Liouville Operators with Transmission Conditions

Mohammad Shahriari*
Department of Mathematics, Faculty of Science, University of Maragheh

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Extended Abstract

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Introduction

Let us consider the boundary value problem involving the differential equation

$$\ell y := -y'' + qy = \lambda y,$$

subject to the boundary conditions

$$U(y) := y'(0) - hy(0) = 0,$$
$$V(y) := y'(\pi) + Hy(\pi) = 0,$$

along with the following discontinuity conditions at a point $a \in (0,\pi)$

$$y(a+0) = a_1 y(a-0), \quad y'(a+0) = a_1^{-1} y'(a-0) + a_2 y(a-0), \tag{1}$$

where $q(x), a, a_1, a_2$ are real, $q \in L^2(0,\pi)$, $a \in (0,\pi)$. Here λ is the spectral independent of x. The coefficients a_1 and a_2 are assumed to be known a priori and fixed. We denote the above boundary value problem by $L = L(q(x); h; H; a)$.

The inverse spectral Sturm-Liouville problem can be regarded as three aspects, e.g., existence, uniqueness and reconstruction of the coefficients given specific properties of eigenvalues and eigenfunctions. Our work concerns uniqueness of potential function and some boundary condition. There are various formulations of the inverse problems and the corresponding uniqueness theorems.

The applications of boundary value problems with discontinuity conditions inside the interval are connected with discontinuous material properties. Inverse problems with a discontinuity ondition inside the interval often appear in mathematics, mechanics, radio electronics, geophysics, and other fields of science and technology.

In this manuscript, we develop the Hochestadt-Lieberman's result for Sturm-Liouville problem when there is a discontinuous condition on the closed interval. We show that the potential function and some coefficients of boundary conditions can be uniquely determined by the value of the potential on some interval and parts of two set of eigenvalues.

Material and methods

In this scheme, first we calculate the asymptotic form of solutions and eigenvalues of the Sturm-Liouville operators with seperated boundary and transmission conditions. Then together with $L = L(q(x); h; H; a)$, we consider a boundary value broblem $\tilde{L} = L(\tilde{q}(x); \tilde{h}; H; a)$ of the same form but a different coefficient $\tilde{q}$ and $\tilde{h}$. So, we prove some inverse problems in the folowing theorems and corollaries.

Results and discussion

In this section, we develop the Hochestadt-Lieberman's result for the Sturm-Liouville problem when there are discontinuous conditions on the closed interval and a fix point on the interval $(0,\pi)$. For this purpose, we have the following theorems and corollaries.

Theorem 1. *Let* $a \in \left(0, \frac{\pi}{2}\right]$ *be a jump point and fix* $b \in \left(0, \frac{\pi}{2}\right]$. *Let* $\lambda_n = \tilde{\lambda}_n$ *for each* $n \in$, $q(x) = \tilde{q}(x)$ *almost everywhere on* $(b,\pi]$. *Then* $q(x) = \tilde{q}(x)$ *almost everywhere on* $[0,\pi]$ *and* $h = \tilde{h}$.

Let $l(n)$ be a subsequence of natural numbers such that

$$l(n) = \frac{n}{\sigma}\left(1+\varepsilon_n\right), \quad 0 < \sigma \le 1,\ \varepsilon_n \to 0, \qquad as\ n \to \infty,$$

and μ_n be the eigenvalues of the problem $L = L\left(q(x);h;H_1;a\right)$, and $\tilde{\mu}_n$ be the eigenvalues of the problem $\tilde{L} = L\left(\tilde{q}(x);\tilde{h};H_1;a\right)$ with the jump conditions (1), such that $H \ne H_1$ and

$$y'(\pi) + H_1 y(\pi) = 0.$$

Theorem 2. *Let* $a \in (0,\pi)$ *be a jump point,* $b \in \left(\frac{\pi}{2},\pi\right]$ *and* $\sigma > \frac{2b}{\pi} - 1$. *Let*

$$\lambda_n = \tilde{\lambda}_n,\ \mu_{l(n)} = \tilde{\mu}_{l(n)} \text{ and } q(x) = \tilde{q}(x) \text{ on } [b,\pi],$$

for each $n \in \mathbb{N}$. *Then* $q(x) = \tilde{q}(x)$ *almost everywhere on* $[0,\pi]$ *and* $h = \tilde{h}$.

Let $m(n)$ be a subsequence of natural numbers such that

$$m(n) = \frac{n}{\sigma_1}\left(1+\varepsilon_{1n}\right), \quad 0 < \sigma_1 \le 1, \quad \varepsilon_{1n} \to 0.$$

Corollary 1. *If* $\lambda_{m(n)} = \tilde{\lambda}_{m(n)}$ *and* $q(x) = \tilde{q}(x)$ *almost everywhere on* $(b,\pi]$ *for* $n \in \mathbb{N}$, $b \le \frac{\pi}{2}$ *and* $\sigma_1 > \frac{2b}{\pi}$ *then* $q(x) = \tilde{q}(x)$ *almost everywhere on* $[0,b]$ *and* $h = \tilde{h}$.

Corollary 2. *Let* $a \in (0,\pi)$ *be a jump point, fix* $b \in \left(\frac{\pi}{2},\pi\right)$, $\sigma_1 > \frac{2b}{\pi}$, *and* $\sigma > \frac{2b}{\pi} - 1$. *Let* $\lambda_{m(n)} = \tilde{\lambda}_{m(n)}$ *and* $\mu_{l(n)} = \tilde{\mu}_{l(n)}$ *for each* $n \in \mathbb{N}$, $q(x) = \tilde{q}(x)$ *almost everywhere on* $(b,\pi]$. *Then* $q(x) = \tilde{q}(x)$ *almost everywhere on* $[0,\pi]$ *and* $h = \tilde{h}$.

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*Corresponding author: shahriari@maragheh.ac.ir