On the Properties of the Arens Regularity of Bounded Bilinear Mappings

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Extended Abstract

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Introduction

Let X, Y and Z be Banach spaces and $f: X \times Y \to Z$ be a bilinear mapping. In 1951 Arens found two extension for f as f^{***} and f^{r***r} from $X^{**} \times Y^{**}$ into Z^{**}. The mapping f^{***} is the unique extension of f such that $x'' \to f^{***}(x'', y'')$ from X^{**} into Z^{**} is $weak^* - weak^*$ continuous for every $y'' \in Y^{**}$, but the mapping $y'' \to f^{***}(x'', y'')$ is not in general $weak^* - weak^*$ continuous from Y^{**} into Z^{**} unless $x'' \in X$. Thus for all $x'' \in X^{**}$ the mapping $y'' \to f^{***}(x'', y'')$ is $weak^* - weak^*$ continuous if and only if f is Arens regular. Regarding A as a Banach $A-bimodule$, the operation $\pi: A \times A \to A$ extends to π^{***} and π^{t***t} defined on $A^{**} \times A^{**}$. These extensions are known, respectively, as the first (left) and the second (right) Arens products, and with each of them, the second dual space A^{**} becomes a Banach algebra.

Material and methods

The constructions of the two Arens multiplications in A^{**} lead us to definition of topological centers for A^{**} with respect to both Arens multiplications. The topological centers of Banach algebras, module actions and applications of them were introduced and discussed in some manuscripts. It is known that the multiplication map of every non-reflexive, C^*-algebra is Arens regular. In this paper, we extend some problems from Banach algebras to the general criterion on module actions and bilinear mapping with some applications in group algebras.

Results and discussion

We will investigate on the Arens regularity of bounded bilinear mappings and we show that a bounded bilinear mapping $f: X \times Y \to Z$ is Arens regular if and only if the linear map $l_2: \mathrm{Y} \to X^*$ with $l_2(y) = f^{r*}(z^*, y)$ is weakly compact, so we prove a theorem that establish the relationships between Arens regularity and weakly compactness properties for any bounded bilinear mappings. We also study on the Arens regularity and weakly compact property of bounded bilinear mapping and we have analogous results to that of Dalse, Ülger and Arikan. For Banach algebras, we establish the relationships between Arens regularity and reflexivity.

Conclusion

The following conclusions were drawn from this research.

- $f^{****}\left(\widehat{Z^*}, X^{**}\right) \subseteq Y^*$ if and only if the bilinear mapping $f: X \times Y \to Z$ is Arens regular.
- A bounded bilinear mapping $\mathrm{f}: \mathrm{X} \times \mathrm{Y} \to \mathrm{Z}$ is Arens regular if and only if the linear map $\mathrm{l}_2: \mathrm{Y} \to \mathrm{X}^*$ with $\mathrm{l}_2(\mathrm{y}) = \mathrm{f}^{\mathrm{r}*}(\mathrm{z}^*, \mathrm{y})$ is weakly compact.
- $f^{\mathrm{r}****\mathrm{r}}\left(Y^{**}, \widehat{Z^*}\right) \subseteq X^*$ if and only if the bilinear mapping $f: X \times Y \to Z$ is Arens regular.
- Assume that $g: X \times X \to X$ has approximate identity. Then g^* is Arens regular if and only if X is reflexive.

Keywords: Arens product, Arens regularity, Banach algebra, bilinear map, second dual, weakly compact.

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