Variational Inequality Problem Over the Set of Common Fixed Points of a Family of Demi-Contractive Mappings

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Extended Abstract

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Introduction

Let C be a nonempty closed convex subset of a real Hilbert space H. Let F : H → H be a monotone operator. The classical variational inequality is formulated as the following problem:

Finding $x^* \in C$ such that $\langle F x^*, y - x^* \rangle \geq 0$, $\forall\, y \in C$. The set of solutions of this problem is denoted by VI (F, C). This problem plays a key role as a suitable models in different fields, such as in mechanics, complementarity problems, economics and optimization theory. Observe that the feasible set C of the variational inequality problem can always be represented as the fixed point set of projection operator. Following this idea, in 2001, Yamada considered the variational inequality problem over the set of fixed points of a nonexpansive mapping. Yamada introduced the so-called hybrid steepest-descent algorithm for solving this problem. In the last decades, many iterative methods have been constructed for solving variational inequalities and their related optimization problems.

Let H and K be real Hilbert spaces, A:H→K be a bounded linear operator and let $\{ C_i \}_{i=1}^{s}$be a family of nonempty closed convex subsets in H and $\{ Q_i \}_{i=1}^{s}$be a family of nonempty closed convex subsets in K. The multiple-set split feasibility problem was introduced by Censor et al. (2005) and is formulated as finding a point x^* with the property: $x^* \in \bigcap_{j=1}^{s} C_i$ and $A(x^*) \in \bigcap_{j=1}^{s} Q_i$. The multiple-set split feasibility problem with s= 1 is known as the split feasibility problem. The split common fixed point problem is formulated as finding a point x^* with the property: $x^* \in$ Fix(U) such that $A(x^*)$ ∈Fix(T); where A:H→K is a bounded linear operator, U:H→H and T:K→K are general operators. It is worth underlining that split common fixed point problem can be regarded as a generalization of the split feasibility problem. The split feasibility problem and the split common fixed point problem have received much attention due to its applications in signal processing, approximation theory, control theory, image reconstruction, with particular progress in intensity-modulated radiation therapy.

Material and methods

Let H be a Hilbert space and let U:H→H be a nonlinear mapping. A point $x \in H$ such that Ux = x is called a fixed point of U. The set of fixed points of nonlinear mapping U shall be denoted by Fix(U). We have the following definitions concerning operator U:H→H. The operator U is called:

•Monotone if

$$\langle U(x) - U(y),\ x - y \rangle \geq 0, \qquad \forall x, y \in H.$$

•Strongly monotone with constant $\beta > 0$, if

$$\langle U(x) - U(y),\ x - y \rangle \geq \beta ||x - y||^2, \qquad \forall x, y \in H.$$

•Inverse strongly monotone with constant β>0, (β−*ism*) if

$$\langle U(x) - U(y),\ x-y \rangle \geq \beta||U(x)-U(y)||^2, \qquad \forall x,y \in H.$$

•Strict pseudo-contractive if there exists μ ∈[0, 1) such that

$$\|Ux-p\|^2 \leq \|x-p\|^2 + \mu\|(x-Ux)-(y-Uy)\|^2,\ \forall x,y \in H.$$

•Demi- contractive if there exists μ ∈[0, 1) such that

$$\|Ux-p\|^2 \leq \|x-p\|^2 + \mu\|x-Ux\|^2,\ \forall x \in H,\ \forall p \in Fix(U).$$

•Lipschitz continuous with constant $L > 0$ if

$$\|U(x) - U(y)\| \leq L\|x-y\|, \quad \forall x,y \in H.$$

If $0 \leq L < 1$, then U is called a contraction. If L=1, then U is called nonexpansive.
In 2000, Moudafi introduced the following so-called viscosity approximation methods:

$$x_{n+1} = a_n f(x_n) + (1-a_n)Tx_n$$

where *f* is contraction and *T* is nonexpansive mapping. He proved that under some appropriate condition imposed on the parameters, the sequence $\{x_n\}$ converges strongly to the unique solution of the variational inequality

$$\langle x^* - f\,x^*, x-x^* \rangle \geq 0, \quad \forall\, x \in Fix(T).$$

In 2001, Yamada introduced the following so-called hybrid steepest descent method:

$$x_{n+1} = (I-\mu\alpha_n F)Tx_n$$

Where *F* is a Lipschitzian continuous and strongly monotone operator and *T* is a nonexpansive operator. Under some appropriate conditions, the sequence $\{x_n\}$ converges strongly to the unique point in *VI*(*F*, Fix(*T*)). In this paper, by using the viscosity iterative method and the hybrid steepest-descent method, we present a new algorithm for solving the variational inequality problem.

Results and discussion

In this paper, by using the viscosity iterative method and the hybrid steepest-descent method, we present a new algorithm for solving the variational inequality problem. The sequence generated by this algorithm is strong convergence to a common element of the set of common zero points of a finite family of inverse strongly monotone operators and the set of common fixed points of a finite family of demi-contractive mappings. Also, we prove that the sequence generated by this algorithm is strong convergence to a solution of a system of variational inequalities over the set of common fixed points of quasi-nonexpansive mappings and strict pseudo-contractive mappings in a Hilbert space. Finally, some applications of this results are present for solving the split common fixed point problem, which entails finding a point which belongs to the set of common fixed points of a finite family of of strict pseudo-contractive mappings in a Hilbert space such that its image under a linear transformation belongs to the set of common fixed points of a finite family of nonexpasive mappings in the image space.

Conclusion

The following conclusions were drawn from this research.

- A new and simple iterative method for solving the variational inequality problem is given. Under mild and standard assumptions, we establish the strong convergence of our algorithm to a solution of a system of variational inequalities over the set of common fixed points of demi- contractive mappings in a Hilbert space.
- Some applications of this results are present for solving the split common fixed point problem,.

Keywords: Variational inequality problem; demi-contractive mappings; fixed point; strict pseudo-contractive mappings.

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