Double-null Operators and the Investigation of Birkhoff's Theorem on Discrete lp Spaces

Ali Bayati Eshkaftaki *
Department of Mathematics, Faculty of Mathematical Sciences,
Shahrekord University, Shahrekord, Iran

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Extended Abstract

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Introduction

Let A be an $m \times n$ matrix. Then A is called non-negative if all of its entries are non-negative. A non-negative matrix A is said to be row (column) stochastic if the summation of each row (column) of A is equal to one. A doubly stochastic matrix is a (square) matrix which is both row and column stochastic. Doubly stochastic matrices, in addition to the matrix theory, are used in several fields, such as combinatorial analysis, inequalities, Schur-convex functions, probabilities, and stochastic (or random) process.

A permutation matrix is a square matrix that has exactly one entry of 1 in each row and each column and zero elsewhere. It is easy to see that every permutation matrix is a doubly stochastic matrix and also $n \times n$ doubly stochastic matrices are a convex subset of all square $n \times n$ matrices.

Birkhoff (1946) showed that doubly stochastic matrices are the convex combination of all permutation matrices. Additionally, the permutation matrices are the extreme points of the set of all doubly stochastic matrices. This result can be extended in various infinite dimensional spaces.

Birkhoff then considered a general question about the extension of this result to infinite matrices. This question later becomes known as Birkhoff's problem 111. This (general) question was considered later in various spaces and in various forms. For example, Isabel (1955) introduced a special space of infinitely many integer matrices called line-finite matrices, and then showed that in this space, the convex hull of the permutation matrices are not equal to the set of all doubly stochastic matrices.

Material and methods

A bounded linear operator $T : l^p(I) \to l^p(I)$ is called

i) doubly stochastic if

$$\forall i \in I,\ \sum_{j \in I} Te_j(i) = 1 \text{ and } \forall j \in I,\ \sum_{i \in I} Te_j(i) = 1.$$

ii) permutation if there exists a bijection $\theta : I \to I$ which satisfies $Te_j = e_{\theta(j)}$ for each $j \in I$.

It is easy to see that every permutation on $l^p(I)$ is a doubly stochastic operator.

Results and discussion

We first introduce double null operators on $l^p(I)$ and find the important properties of them. Then Based on these operators, we investigate the Birkhoff theorem in infinite dimensions.

Keywords: Doubly stochastic operator, Double-null operator, Birkhoff's problem 111, Extreme points

*Corresponding author: Bayati.ali@sku.ac.ir