

Wavelet Sets on Locally Compact Abelian Groups

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Extended Abstract

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Introduction

An orthonormal wavelet is a square-integrable function whose translates and dilates form an orthonormal basis for the Hilbert space $L^2(\mathbb{R})$. That is, given the unitary operators of translation

$T_n f(x) = f(x - n)$ for $n \in \mathbb{Z}$ and dilation $Df(x) = \sqrt{2}f(2x)$, we call $\psi \in L^2(\mathbb{R})$ an orthonormal wavelet if the set

$$\{D^j T_n \psi : j, n \in \mathbb{Z}\} = \{2^{j/2} \psi(2^j \cdot - n) : j, n \in \mathbb{Z}\}$$

is an orthonormal basis for $L^2(\mathbb{R})$. This definition was later generalized to higher dimensions

and to allow for other dilation and translation sets; let Hilbert space $L^2(\mathbb{R}^n)$ and an $n \times n$ expansive matrix A (i.e. a matrix with eigenvalues bigger than 1) with integer entries, then dilation operator D_A is given by $D_A f(x) = \sqrt{|A|}f(Ax)$ and the translation operator T_n is given

by $T_n f(x) = f(x - n)$ for $n \in \mathbb{Z}^n$. A finite set $\Psi = \{\psi_1, \psi_2, \dots, \psi_l\} \in L^2(\mathbb{R}^n)$ is called multiwavelet if the set

$$\{D_A^j T_n \psi_i : 1 \leq i \leq l, n \in \mathbb{Z}^n, j \in \mathbb{Z}\},$$

is an orthogonal basis for $L^2(\mathbb{R}^n)$.

Central to the theory of wavelets is the concept of a multiresolution analysis, abbreviated

MRA. There is much overlaps between wavelet analysis and Fourier analysis. Indeed, wavelets can be thought of as non-trigonometric Fourier series. Thus, Fourier analysis is used as a tool to investigate properties of wavelets.

Another concept is wavelet set. The term wavelet set was coined by Dai and Larson in the late 90s to describe a set W such that χ_W , the characteristic function of W , is the Fourier transform of an orthonormal wavelet on $L^2(\mathbb{R})$. At about the same time as the Dai and Larson paper, Fang and Wang first used the term MSF wavelet (minimally supported frequency wavelet) to describe wavelets whose Fourier transforms are supported on sets of the smallest possible measure. The importance of MSF wavelets as a source of examples and counterexamples has continued throughout wavelet history. A famous example due to Journé first showed that not all wavelets have an associated structure multiresolution analysis (MRA). The discovery of a non-MRA wavelet gave an important push to the development of more general structures such as frame multiresolution analyses (FMRAs) and generalized

multiresolution analyses (GMRA's). In this paper we generalize wavelets and wavelet sets on locally compact abelian group G with uniform lattice.

Material and methods

In this paper, we investigate wavelet sets on locally compact abelian groups with uniform lattice, where a uniform lattice H in LCA group G is a discrete subgroup of G such that the quotient group G/H is compact. So we review some basic facts from the theory of LCA groups and harmonic analysis. Then we define wavelet sets on these groups and characterize them by using of Fourier transform and multiresolution analysis.

Results and discussion

We extend theory of wavelet sets on locally compact abelian groups with uniform lattice. This is a generalization of wavelet sets on Euclidean space. We characterize wavelet sets by using of Fourier transform and multiresolution analysis. Also, we define generalized scaling sets and dimension functions on locally compact abelian groups and verify its relations with wavelet sets. Dimension functions for MSF wavelets are described by generalized scaling sets.

In the setting of LCA groups, we define translation congruent and show wavelet sets are translation congruent, so we can define a map on G that is measurable, measure preserving and bijection.

Conclusion

The following conclusions were drawn from this research.

- Wavelet sets on locally compact groups by uniform lattice can be defined. This is a generalization of wavelet sets on Euclidean space.
- Characterization of wavelet sets on LCA groups can be done in different ways. A method is to use Fourier transform and translation congruent. Another way is generalized scaling set and dimension function.
- As an example Cantor dyadic group is a non-trivial example that satisfies in the theory of wavelet sets on locally compact groups by uniform lattice. We find wavelet set and generalized scaling set for this group and show related wavelet is MRA wavelet.

Keywords: wavelet; locally compact abelian group; wavelet set; multiresolution analysis; generalized scaling sets.

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