On Subadditivity of Functions on Positive Operators Without Operator Monotonicity and Convexity

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Extended Abstract

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Introduction

It is known that if h is a non-negative concave function on the interval $[0,\infty)$, then $\frac{h(x)}{x}$ is decreasing and hence, h is subadditive. In [6], it is proved that if f is a non-negative operator monotone function on $[0,\infty)$, then

$$f(A+B) \leq f(A)+f(B),$$

where, A, B are positive operators on a Hilbert space $\mathcal{H}$ and $AB+BA \geq 0$. In [8], a similar result is proved for finitely many operators by a different method. Moreover, the inequality

$$g(A_1+\cdots+A_n) \geq g(A_1)+\cdots+g(A_n),$$

is obtained for every operator convex function g on $[0,\infty)$ with $g(0) \leq 0$ and positive operators $A_1,\ldots,A_n$.

In this paper, we investigate the subadditivity of functions on positive operators without operator monotonicity and operator convexity.

Material and methods

To obtain the subadditivity of functions on positive operators without operator monotonicity and operator convexity, first we apply a theorem which gives a type of operator monotonicity for convex (not necessarily operator convex) functions [2]. Then, to complete the proof of the main result, we use a version of Jensen's operator inequality which is proved for convex functions without any assumption of operator convexity [1,4].

Results and discussion

Let operator A with bounds m_1, M_1 and operator B with bounds m_2, M_2 be positive operators on a Hilbert space $\mathcal{H}$ satisfying $0 \leq AB+BA$. Let

$$E=(A+B)^{-\frac{1}{2}}(A^2+B^2)(A+B)^{-\frac{1}{2}}.$$

Suppose that

$$(m_E, M_E) \cap [m_i, M_i] = \emptyset \quad (i=1,2),$$

where, m_E and M_E are bounds of operator E. Then, for every continuous function $g:(0,\infty) \to \mathbb{R}^+$ for which the function $f(t)=\frac{g(t)}{t}$ is convex and decreasing, we have

$$g(A+B) \leq c(m,M,f)(g(A)+g(B)),$$

where, m and M are bounds of operator $A+B$ and

$$c(m,M,f) := \max_{m \leq t \leq M} \left\{ \frac{\frac{f(M)-f(m)}{M-m}t+\frac{Mf(m)-mf(M)}{M-m}}{f(t)} \right\}.$$

This theorem is the main result of the paper. Also, applying a result of [1], if we replace the condition

$$(m_E, M_E) \cap [m_i, M_i] = \emptyset \quad (i = 1,2),$$

by the condition that the open interval (m_E, M_E) does not intersect the spectrums of operators A and B, then the same result is obtained.

Conclusion

A version of subadditivity of convex decreasing functions on positive operators without any assumption of operator monotonicity and operator convexity is obtained.

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