Topological Invariant of Integrable Hamiltonian System on Cone Located in a Potential Field

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Extended Abstract

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Introduction

The theory of topological classification of integrable Hamiltonian systems with two degrees of freedom due to Fomenko and his school. On the basis of this theory we give a topological Liouville classification of the integrable Hamiltonian systems with two degrees of freedom. Essentially, to an integrable system with two degrees of freedom which is restricted to a nonsingular 3-dimensional iso-energy manifold. Fomenko's theory ascribes in an effective way a certain discrete invariant which has the structure of a graph with numerical marks. This invariant, which is called the marked molecule or the Fomenko-Zieschang invariant, gives a full description (up to Liouville equivalence) of the Liouville foliation for the system.
The topological classification of integrable Hamiltonian systems corresponding to the Liouville equivalence in potential fields on surfaces of revolution for surfaces that is diffeomorphic with 2-dimensional sphere, contains a wide classes of mechanical systems that describes the motion of a particle on a 2-dimensional sphere with revolution metric, which has been studied.
In this paper, the topology of non-singular iso-energy surfaces for a Hamiltonian system with two degrees of freedom on a cone located in a potential field is described. Also, the method of finding the topological invariant of integrable Hamiltonian systems is extended from compact case to non-compact rotating surfaces.

Material and methods

This paper describes the topology of non-singular iso-energy surfaces for a Hamiltonian system with two degrees of freedom on a cone located in a potential field. The motion on the cone defined by function $f(r) = r$ in the field of $V(r) = ar$.
Consider the equation of a general cone as $z = a\sqrt{x^2 + y^2}$. In polar coordinate, we have Hamiltonian as follows:

$$H = \frac{1}{2(1+a^2)} p_r^2 + \frac{1}{2(r^2)} p_\theta^2 + ar$$

where (r, θ) are the coordinates of the point and (p_r, p_θ) are the momenta. The motion of the point on cone described by the Hamiltonian equations

$$\dot{p_r} = \frac{\partial H}{\partial r}, \quad \dot{p_\theta} = -\frac{\partial H}{\partial \theta}.$$

We demonstrate a method based on the investigation of the bifurcation diagram of the momentum mapping. The general idea is as follows:
Step 1. We calculate the bifurcation diagram of the momentum mapping $\phi: M^4 \longrightarrow R^2(h,k)$. As a result we obtain (in general case) some curve with singularities in the two-plane R^2.

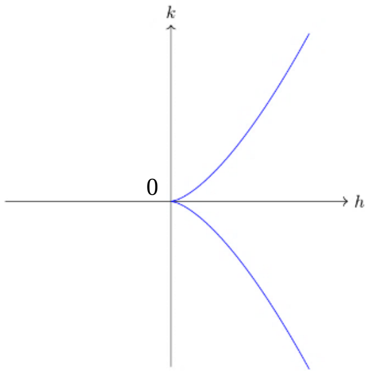
Step 2. We consider the line $\ell = \{H = h = \text{const}\}$ on the plane (H, f) and the points where the line intersects Σ. The pre-image of the line $\phi^{-1}(\ell)$ is a three-manifold Q_h^3 in M.
Step 3. We investigate all points of intersection $\Sigma \cap \ell$ and calculate the Hessian of f for the corresponding critical sub-manifold in Q_h^3.
Step 4. We can describe the type of surgery of Liouville tori on these critical sub-manifolds based on the main theorem of classification of all possible transformations of Liouville tori.
We extend this method by using effective potential to calculate the number of connected components of iso-energy surface in non-compact cases.

Main Results

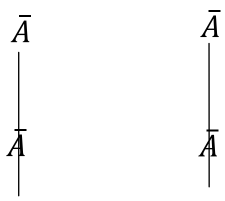
- **Proposition 1.** For pair $((f(r), V(r))$ the system with Hamiltonian H on a cone is completely Liouville integrable.
- **Lemma 1.** Defined mechanical system on cotangent bundle of cone has a 2-parameter family of rank 1 singular points $(r, \theta, 0, k(r))$ where $k(r) = \pm\sqrt{r^3 a}$ and has no points of rank 0. The image under the momentum map of the family of the singular points of rank 1 has a smooth parameterization of the form

$$h(r) = \frac{3}{2} ar \quad , \quad k(r) = \pm\sqrt{r^3 a}.$$

where r ranges over the open set I. The bifurcation diagram of the system is symmetric relative to the Oh axis.



- **Lemma 2.** Let $Q \subseteq Q_h^3$ be a connected component of nonsingular iso-energy surface, then $K|_{Q_h^3}$ is a Bott function and the rough molecule corresponding to Q contains atoms kind of $\bar{A}$.



Keywords: Hamiltonian System, Iso-energy Surfaces, Fomenko-Zieschang invariant, Potential field.

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