Mathematical Researches (Sci. Kharazmi University) Vol. 7, No. 1 Spring 2021

Extension Functors of Generalized Local Cohomology Modules

Alireza Vahidi*, Faisal Hassani, Elham Hoseinzade
Department of Mathematics, Payame Noor University, Tehran

Received: 2018/10/17 Accepted: 2019/08/05

Extended Abstract

Paper pages (151-164)

Introduction

Throughout this paper, R is a commutative Noetherian ring with non-zero identity, $\mathfrak{a}$ is an ideal of R, M is a finitely generated R-module, and X is an arbitrary R-module which is not necessarily finitely generated.

Let L be a finitely generated R-module. Grothendieck, in [11], conjectured that $\mathrm{Hom}_R\left(R/\mathfrak{a}, H^j_{\mathfrak{a}}(L)\right)$ is finitely generated for all j. In [12], Hartshorne gave a counter-example and raised the question whether $\mathrm{Ext}^i_R\left(R/\mathfrak{a}, H^j_{\mathfrak{a}}(L)\right)$ is finitely generated for all i and j. The jth generalized local cohomology module of M and X with respect to $\mathfrak{a}$,

$$H^j_{\mathfrak{a}}(M, X) \cong \varinjlim_{n\in\mathbb{N}} \mathrm{Ext}^j_R(M/\mathfrak{a}^n M, X),$$

was introduced by Herzog in [14]. It is clear that $H^j_{\mathfrak{a}}(R, X)$ is just the ordinary local cohomology module $H^j_{\mathfrak{a}}(X)$ of X with respect to $\mathfrak{a}$. As a generalization of Hartshorne's question, we have the following question for generalized local cohomology modules (see [25, Question 2.7]).

Question. When is $\mathrm{Ext}^i_R\left(R/\mathfrak{a}, H^j_{\mathfrak{a}}(M, L)\right)$ finitely generated for all i and j?

In this paper, we study $\mathrm{Ext}^i_R\left(N, H^j_{\mathfrak{a}}(M, X)\right)$ in general for a finitely generated R-module N and an arbitrary R-module X.

Material and methods

The main tool used in the proofs of the main results of this paper is the spectral sequences.

Results and discussion

We present some technical results (Lemma 2.1 and Theorems 2.2, 2.9, and 2.14) which show that, in certain situation, for non-negative integers s, t, s′, and t′ with $s + t = s' + t'$, $\mathrm{Ext}^s_R\left(N, H^t_{\mathfrak{a}}(M, X)\right) \cong H^{s'}_{\mathfrak{a}}\left(\mathrm{Tor}^R_{t'}(N, M), X\right)$ and the R-modules $\mathrm{Ext}^s_R\left(N, H^t_{\mathfrak{a}}(M, X)\right)$ and $H^s_{\mathfrak{a}}(\mathrm{Tor}^R_t(N, M), X)$ are in a Serre subcategory of the category of R-modules (i.e. the class of R-modules which is closed under taking submodules, quotients, and extensions).

Conclusion

We apply the main results of this paper to some Serre subcategories (e.g. the class of zero R-modules and the class of finitely generated R-modules) and deduce some properties of generalized local cohomology modules. In Corollaries 3.1-3.3, we present some upper bounds for the injective dimension and the Bass numbers of generalized local cohomology modules. We study cofiniteness and minimaxness of generalized local cohomology modules in Corollaries 3.4-3.8. Recall that, an R-module X is said to be $\mathfrak{a}$-cofinite (resp. minimax) if $\mathrm{Supp}_R(X) \subseteq$

$\mathrm{Var}(\mathfrak{a})$ and $\mathrm{Ext}_R^i(R/\mathfrak{a}, X)$ is finitely generated for all i [12] (resp. there is a finitely generated submodule X' of X such that X/X' is Artinian [26]) where
$\mathrm{Var}(\mathfrak{a}) = \{\mathfrak{p} \in \mathrm{Spec}(R) | \mathfrak{p} \supseteq \mathfrak{a}\}$. We show that if $\mathrm{Ext}_R^i(R/\mathfrak{a}, X)$ is finitely generated for all $i \leq t$ and $H_\mathfrak{a}^i(M, X)$ is minimax for all $i < t$, then $H_\mathfrak{a}^i(M, X)$ is $\mathfrak{a}$-cofinite for all $i < t$ and $\mathrm{Hom}_R(R/\mathfrak{a}, H_\mathfrak{a}^t(M, X))$ is finitely generated (Corollary 3.5). We prove that if $\mathrm{Ext}_R^i(R/\mathfrak{a}, X)$ is finitely generated for all $i \leq \mathrm{ara}(\mathfrak{a})$, where $\mathrm{ara}(\mathfrak{a})$ is the arithmetic rank of $\mathfrak{a}$, and $H_\mathfrak{a}^i(M, X)$ is $\mathfrak{a}$-cofinite for all $i \neq t$, then $H_\mathfrak{a}^t(M, X)$ is also an $\mathfrak{a}$-cofinite R-module (Corollary 3.6). We show that if R is local, $\dim(R/\mathfrak{a}) \leq 2$, and $\mathrm{Ext}_R^i(R/\mathfrak{a}, X)$ is finitely generated for all $i \leq t + 1$, then $H_\mathfrak{a}^i(M, X)$ is $\mathfrak{a}$-cofinite for all $i < t$ if and only if $\mathrm{Hom}_R\left(R/\mathfrak{a}, H_\mathfrak{a}^i(M, X)\right)$ is finitely generated for all $i \leq t$ (Corollary 3.7). We also prove that if R is local, $\dim(R/\mathfrak{a}) \leq 2$, $\mathrm{Ext}_R^i(R/\mathfrak{a}, X)$ is finitely generated for all i, and $H_\mathfrak{a}^{2i}(M, X)$ (or $H_\mathfrak{a}^{2i+1}(M, X)$) is $\mathfrak{a}$-cofinite for all i, then $H_\mathfrak{a}^i(M, X)$ is $\mathfrak{a}$-cofinite for all i (Corollary 3.8). In Corollary 3.9, we state the weakest possible conditions which yield the finiteness of associated prime ideals of generalized local cohomology modules. Note that, one can apply the main results of this paper to other Serre subcategories to deduce more properties of generalized local cohomology modules.

Keywords: Associated prime ideals, Bass numbers; Cofinite modules, Extension functors, Generalized local cohomology modules, Injective dimensions, Minimax modules.

*Corresponding author: vahidi.ar@pnu.ac.ir