Almost Uniserial Modules

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Extended Abstract

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Introduction

In this paper, all rings have identity elements and all modules are unitary modules. A uniserial ring is a ring whose lattice of ideals is linearly ordered under inclusion. Note that commutative uniserial rings are also known as valuation rings. An R-module M is called uniserial if its submodules are linearly ordered by inclusion. Also an R-module M is called serial if it is a direct sum of uniserial modules. Thus a uniserial ring is a ring which is uniserial as a module. Also a ring R is called serial if it is a serial R-module.

The study of serial ring has a long history. Maybe the first important contribution in this direction is due to Köthe [7]. Köthe showed that the modules over an Artinian principal ideal rings (which are a special case of serial rings) are direct sums of cyclic submodules. Later, Cohen and Kaplansky determined that, for a commutative ring R, all R-modules are direct sums of cyclic submodules if and only if R is an Artinian principal ideal ring [4]. Nakayama showed that if R is an Artinian serial ring, then all R-modules are direct sums of cyclic submodules, and that the converse is not true [8]. Asano ([1,2]) proved that a commutative ring R is Artinian serial if and only if R is an Artinian principal ideal ring. A partial list of important contributors to the theory of serial rings includes the mathematicians Äsano, Cohen, Goldie, Kaplansky, Nakayama, Warfield (for references for each author see [1,2,4–9]).

The class of almost uniserial rings and modules are introduced by M. Behboodi and S. Roointan-Isfahani as a straightforward common generalization of uniserial rings and principal ideal domains [3]. A ring R is called almost uniserial if any two non-isomorphic ideals of R are linearly ordered by inclusion. We note that each uniserial ring is almost uniserial, but the converse is not true in general. For instant $\mathbb{Z}$ is not a uniserial ring but is almost uniserial ring.

Fitting ideals

Let R be a commutative Noetherian ring with identity and M be a finitely generated R-module. Let $F \xrightarrow{\varphi} G \xrightarrow{\psi} M \longrightarrow 0$, be a free presentation of M with G a free module of rank r. Let $A \in M_{r\times s}(R)$ be a matrix presentation of φ and $I_j(\varphi)$ be an ideal of R generated by the minors of size j of matrix A. By convention, the determinant of the 0×0 matrix is 1. In general, we set $I_j(\varphi)=R$ if $j \le 0$.

It is known that, the ideals $I_{r-j}(\varphi)$ are independent of the choice of free presentation of M. So we define the j th Fitting ideal of M to be the ideal $Fitt_j(M)=I_{r-j}(\varphi)$. The most important Fitting ideal of M is the first of the $Fitt_j(M)$ that is nonzero. We shall denote this Fitting ideal by $I(M)$. Thus $I(M)=I_{rank(\varphi)}(\varphi)$. It follows from the definition that $Fitt_i(M) \subseteq Fitt_{i+1}(M)$. for every i.

Fitting ideals are strong tools to characterize modules and to recognize some properties of them.
In this paper we investigate some properties of almost uniserial modules and construct a torsion almost uniserial R -module whose first nonzero Fitting ideal is a product of maximal ideals. Also we characterize all torsion-free almost uniserial module over an integral domain and a unique factorization domain.

Main results

The following results are shown in this paper.

1) Let R be a Noetherian ring and M be a finitely generated almost uniserial R -module. Then M is a torsion module or is a torsion-free module.
2) Let $0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$ be an exact sequence of R -modules. If B is an almost uniserial module, then A and C are almost uniserial.
3) Let (R,P) be a Noetherian local and commutative ring. Let M be a finitely generated almost uniserial module such that $I(M)=P^{\alpha}$, for some $\alpha \in \mathbb{N}$. If M is a torsion module, then $M \cong \frac{R}{P^{\alpha}}$ or $M \cong \frac{R}{P} \oplus \frac{R}{P}$ or $P^{n}M \cong (\frac{R}{P})^{k}$, $\exists k=1,2$, for some $n \leq \alpha - 1$.
4) Let R be a Noetherian integral domain and M be a finitely generated almost uniserial R -module. If M is torsion-free, then $M \cong \frac{R^2}{<\begin{pmatrix} a \\ b \end{pmatrix}>}$ or $M \cong \frac{R^2}{\langle \begin{pmatrix} a & c \\ b & d \end{pmatrix} \rangle}$, for some $a,b,c,d \in R$.
5) Let R be a Noetherian unique factorization domain and M be a finitely generated almost uniserial module. If M is torsion-free, then $M \cong \frac{R^2}{\langle \begin{pmatrix} a \\ b \end{pmatrix} \rangle}$, for some $a,b \in R$.

Keywords: Almost uniserial module, Torsionfree module, Fitting ideals, Unique factorization domain

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