Convexity and Geodesic Metric Spaces

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Abstract

In this paper, we first present a preliminary study on metric segments and geodesics in metric spaces. Then we recall the concept of d-convexity of sets and functions in the sense of Menger and study some properties of d-convex sets and d-convex functions as well as extreme points and faces of d-convex sets in normed spaces. Finally we study the continuity of d-convex functions in geodesic metric spaces.

Keyword: Geodesic, Metric segment, d-convex set, Metric convex hull, d-convex function, Extreme point, Continuity.
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Extended Abstract

Introduction

Convexity plays an essential role in many branches of mathematics like optimization, differential equations, nonlinear and variational analysis. This concept is traditionally defined in linear spaces. A subset C of a linear space X is called convex if for any two points x and y of C, the usual segment $\{(1-t)x+ty|0\leq t\leq 1\}$ is contained in C. A real-valued function f on a convex subset C is called convex if

$$f\big((1-t)x\ +ty\big)\leq\ (1-t)f(x)+tf(y),\qquad x,y\in C,\qquad t\in[0,1].$$

To define the concept of convexity in a general metric space (X,d), first the notion of metric segment is defined. A metric segment between two points x and y is defined as the set

$$[x,y]\coloneqq\{z\in X|d(x,z)+d(z,y)=d(x,y)\}.$$

A subset C of X is called convex if for any two points x and y in C we have $[x,y]\cap C-\{x,y\}\neq\emptyset$. Here we use a stronger definition of metric convexity. A subset $C\subseteq X$ is called $d-$convex if $[x,y]\subseteq C$, for each $x,y\in C$.

A real valued function f on a d-convex subset C is called d-convex if and only if for any two different points x_1 and x_2 in C and $x_0\in[x_1,x_2]$,

$$f(x_0)\leq\frac{d(x_0,x_2)}{d(x_1,x_2)}f(x_1)+\frac{d(x_0,x_1)}{d(x_1,x_2)}f(x_2),$$

and it is called strongly d-convex if

$$f(x_0)\leq\frac{d(x_0,x_2)}{d(x_1,x_2)}f(x_1)+\frac{d(x_0,x_1)}{d(x_1,x_2)}f(x_2)-d(x_0,x_1)d(x_0,x_2).$$

Definition of d-convex sets, d-convex functions and strongly d-convex functions are natural extensions of convexity in linear spaces. It is easy to see that d-convexity of sets and functions implies convexity in any normed linear space, but the converse holds only in strictly convex normed linear spaces.

Metric linear spaces have more general structures than normed linear spaces, in which both definitions of convexity hold. Some geometric notions of normed spaces like strict convexity and pseudo strict convexity are defined in metric linear spaces similarly. It is proved that in each normed space, strict convexity is equivalent to pseudo-strict convexity. Here we show that in metric linear spaces with an invariant metric, pseudo strict convexity is equivalent to the fact that every metric segment $[x, y]$ is contained in the usual segment connecting x to y.

One of the more important arguments in convexity is extreme points of convex sets. We study the extreme points and faces of unit balls and metric segments in metric spaces and consider their relation with geodesics.

In the end of the paper, we follow continuity properties of convex functions in metric spaces and establish the relation between convex functions with continuity, locally bondedness and locally Lipchitz of functions.

The paper is organized as follows. In Section 2, we study some elementary properties of metric segments and a partial order on them. Section 3, is devoted to the study of geodesics in metric spaces. We study existence and uniqueness of geodesics in a metric space under some appropriate conditions on the metric space. We also study relations between a geodesic curve connecting two given points and the partial order (defined in Section 2) in the metric segment generated by the points. In Section 4, we study balls and metric segments from convexity point of view. In section 5, some continuity properties of convex functions in metric spaces are studied.

Material and methods

The concept of convexity in linear spaces is generalized to metric spaces. Convex subsets of a metric space and real valued convex functions on these spaces are defined and studied. Also a new sight to the study of geodesics in metric spaces is given by using a partial order relation.

Results and discussion

Among other results, the main results of this paper may be as follows:

1. The relation between extreme points of the unit ball in a metric space and uniqueness of geodesics in the space is determined.
2. The relation between faces and extreme points of metric segments is specified.
3. It is proved that many continuity properties of convex functions on normed linear spaces are similarly hold for convex functions on metric spaces.

Conclusion

Convexity is usually defined in linear spaces. Here, this concept is defined and studied in metric spaces in the absence of linear structures. This concept may also be studied in discrete metric spaces, non-geodesic metric spaces or even in non-unique geodesic metric spaces. These studies can be continued in general metric spaces those clearly have no either linear nor (continuous) geodesic structures, such as graphs or any other discrete structures.

Keywords: Metric segment; Metric convex set; Metric convex function; Geodesic; Extreme point; Continuity.

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