Generalized Local Operators Between Function Modules

Fereshteh Sady*[1], Masoumeh Najafi Tavani[2]

1. Department of Pure Mathematics, Faculty of Mathematical Sciences, Tarbiat Modares University, Tehran
2. Department of Mathematics, Faculty of Mathematical Sciences, Islamic Azad University, Islamshahr Branch, Tehran

Received: 2018/09/24 Accepted: 201909/30

Extended Abstract

Paper pages (91-100)

Introduction

Let X be a compact Hausdorff space and $A(X)$ and $B(X)$ be subspaces of $C(X)$ such that $A(X) \subseteq B(X)$. A linear operator $T: A(X) \longrightarrow B(X)$ is called a local map, if $Tf \cdot g = 0$ for any $f \in A(X)$ and $g \in B(X)$ with $f \cdot g = 0$. For instance, all multiplication operators and the differential operator are local maps. Clearly any local map is a separating map, in the sense that $f \cdot g = 0$ implies $Tf \cdot Tg = 0$. In general, a linear operator between subspaces of vector-valued continuous functions is called a separating map if it sends any pair of functions with disjoint cozeros to a pair of such functions. Linear separating maps between various spaces of scalar or vector-valued spaces of functions have been intensively studied by many authors such as K. Jarosz, J. Araujo, J.J. Font, S. Hernandez, A. Jimenez-Vargas and L. Dubarbie. These spaces include the spaces of all continuous functions and its certain subspaces such as Lipschitz spaces of functions, absolutely continuous functions and continuously differentiable functions. In most cases such maps have representations as generalized weighted composition operators.

In this paper we first introduce the notion of (additive) local maps with respect to n additive maps between vector-valued spaces of functions having $A(X)$-module structure for some regular Banach function algebra $A(X)$ on a compact Hausdorff space X. Then we give a description of the given $n+1$ maps as generalized weighted composition operator

Main Results

Throughout this paper X is a compact Hausdorff space and E is a real or complex normed space. The space of all E-valued continuous functions on X will be denoted by $C(X,E)$ and the supremum norm on $C(X,E)$ is denoted by $\|\cdot\|_\infty$. For the case that $E = \mathbb{C}$ we use the notation $C(X)$ instead of $C(X,E)$. For a function $f \in C(X,E)$ we denote the cozero set of f by $\mathrm{coz}(f)$. A subalgebra A of $C(X)$ which contains the constant functions and separates the points of X is called a Banach function algebra on X if it is a Banach algebra under some norm.

Definition 1. Let $A(X)$ be a regular Banach function algebra on X and $A(X,E)$ be a subspace of $C(X,E)$ containing constants such that $A(X) \cdot A(X,E) \subseteq A(X,E)$. Let $n \in \mathbb{N}$ and $T_1, T_2, \ldots T_n: A(X) \longrightarrow C(X)$ and also $S: A(X,E) \longrightarrow C(Y,F)$ be additive maps where Y is a compact Hausdorff space and F is a normed space. We say that S is local with respect to $T_1, \ldots, T_n$ if for each $f_1, \ldots, f_n \in A(X)$ and $g \in A(X,E)$ with $f_1 f_2 \cdots f_n g = 0$ we have

$$T_1 f_1 \cdot T_2 f_2 \cdots T_n f_n \cdot Sg = 0.$$

We note that in particular cases, this definition coincides with the definition of local maps and also separating maps.

Subspaces of vector-valued continuous functions that we will consider in this paper are required to satisfy a property introduced in the next definition.

Definition 2. Let Y be a compact Hausdorff space and F be a real or complex normed space. We say that a subspace $A(Y, F)$ of $C(Y, F)$ has property (P) if for each $y \in Y$ and $f \in A(Y, F)$ with $f(y) = 0$, there exists a sequence $\{f_n\}$ in $A(Y, F)$ such that each f_n vanishes on a neighbourhood of y and, moreover, $\|f_n - f\|_\infty \to 0$.

Here are some examples of subspaces having property (P).

Examples. (i) For a compact Hausdorff space X and a normed space E, the space $C(X, E)$ itself has property (P).

(ii) For any compact metric space (X, d) and a normed space E, the space $\mathrm{Lip}^\alpha(X, E)$ of all E-valued Lipschitz functions of order $\alpha \in (0,1]$ on X has property (P).

(iii) For $I = [0,1]$, and a normed space E, the space $AC(I, E)$ of all E-valued absolutely continuous functions on I has property (P).

Let $A(X)$ and $A(X, E)$ be subspaces of $C(X)$ and $C(X, E)$, respectively. For additive maps $T_1, \dots T_n, : A(X) \longrightarrow C(X)$ and $S: A(X, E) \longrightarrow A(X, E)$ we consider the following subset of X:

$$X_1 = \{y \in X: \prod_{i=1}^{n} T_i 1(y) \neq 0\} \cap \left(\cup_{g \in A(X,E)} \mathrm{coz}(Sg)\right).$$

It is clear that $X_1 = X$ if the range of S contains a nonzero constant function and each T_k is unital, that is $T_k(1) = 1$ for $k = 1, \dots, n$.

The main result of this paper is as follows:

Theorem 3. Let $A(X)$ be a regular Banach function algebra on its maximal ideal space X and let $A(X, E)$ be a subspace of $C(X, E)$ containing constant functions such that $A(X) \cdot A(X, E) \subseteq A(X, E)$. Assume, furthermore, that $A(X)$ and $A(X, E)$ have property (P). If $T_1, \dots T_n, : A(X) \longrightarrow C(X)$ and $S: A(X, E) \longrightarrow A(X, E)$ are additive continuous maps such that S is local with respect to $T_1, \dots, T_n$, then there exist a continuous map $\varphi: X_1 \longrightarrow X$ and a family $\{J_y: y \in X_1\}$ of continuous linear operators on E such that for all $y \in X_1$ we have

$$T_k f(y) = T_k(f(\varphi(y)))(y) \qquad (f \in A(X), k = 1, \dots, n),$$

$$Sg(y) = J_y\left(g(\varphi(y))\right) \qquad \left(g \in A(X, E)\right).$$

In particular, for each $k = 1, \dots, n$ and $y \in X_1$ we have

$$T_k f(y) = \mathrm{Re}(f(\varphi(y)))\, T_k 1(y) + \mathrm{Im}(f(\varphi(y)))\, T_k i(y) \qquad \left(f \in A(X)\right)$$

Keywords: local map, Separating map, Banach function algebra, Lipschitz function.

*Corresponding author sady@modares.ac.ir