A Note on Hilbert Modules

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Extended Abstract

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Introduction and preliminaries

Hilbert C∗-modules were firrst introduced in the work of I. Kaplansky.Hilbert C*-modules are the natural generalization that of Hilbert spaces arising by replacing of the field of scalars C by a C∗-algebra. Let us recall some basic facts about the Hilbert C∗-modules.

Let A be a C∗-algebra. An right inner product A-module is a linear space X which is a right A-module (with compatible scalar multiplication: λ(x.a) =(λx).a = x.(λa) for x ∈ X, a ∈ A, λ ∈ C), together with a map (x, y)→ $\langle x, y\rangle_X$: X × X → A such that for all x, y, z ∈ X, a ∈ A, α, β ∈ C

(i) $\langle x, \alpha y + \beta z\rangle_X$ = α$\langle x, y\rangle_X$+ β$\langle x, z\rangle_X$;

(ii) $\langle x, y.a\rangle_X$ =$\langle x, y\rangle_X$ a;

(iii) $\langle y, x\rangle_X$ = $\langle x, y\rangle_X^*$;

(iv) $\langle x, x\rangle_X$≥ 0 ; if $\langle x, x\rangle_X$= 0 then x = 0 .

A right pre-Hilbert A-module X is called a right Hilbert A-module if it is complete with respect to the norm ‖x‖=‖ $\langle x, x\rangle_X$ ‖$^{\frac{1}{2}}$. X is said to be full if the linear span of the set {$\langle x, y\rangle_X$: x, y ∈ X} is dense in A. One interesting example of full right Hilbert C∗-modules is any C∗-algebra A as a right Hilbert A-module via $\langle a, b\rangle_A$= a∗b (a, b ∈ A). Likewise, a left Hilbert A-module with an A-valued inner product $_X\langle .,.\rangle$ can be defined.

Let X be a right Hilbert A-module, we define L(X) to be the set of allmaps T : X → X for which there is a map T* : X → X such that $\langle Tx, y\rangle_X$ = $\langle x, T*y\rangle_X$ (x, y ∈ X). The map T* is called the adjoint of T and it is easy to see that T must be bounded and A-linear. We call L(X) the set of adjointable operators on X and it is a C*-algebra. For x, y ∈ X, define the operator $\theta_{x,y}$ on X by $\theta_{x,y}(z)$ = x.$\langle y, z\rangle_X$ (z ∈ X). Denote by K(X) the closed linear span of {$\theta_{x,y}$: x, y ∈ X}, then K(X) is a closed two sided ideal in L(X). Elements of K(X) are often referred to as compact operators.

Let X be an A-bimodule. X is said to be a Hilbert A-bimodule, when X is a left and right Hilbert A-module and satisfies the relation $_X\langle x, y\rangle$.z = x.$\langle y, z\rangle_X$. We notice that ‖ $_X\langle x, x\rangle$ ‖=‖$\langle x, x\rangle_X$‖ for all x ∈ X. In this paper a derivation of an algebra A is a linear mapping D from A into itself such that D(ab) = D(a)b+aD(b) for all a, b∈A. For a fixed b ∈ A, the mapping a → ba− ab is clearly a derivation, which is called an inner derivation implemented by b.

Results and discussion

Let A be a C*-algebra. Let X be a left Hilbert A-module, and let e be an arbitrary element in X with

‖e‖ = 1. Then the map $\pi_e: X \times X \to X$ defined by $\pi_e(x, y) = {}_X< x, e >. y$ is a product on X and X together with this product is a Banach algebra. We denote this Banach algebra by (X,π_e).

In the second section of this paper, we first prove that for a Full right Hilbert A-module X, a d-derivation δ on X is a derivation on (X,π_e) if and only if $d(< e,x >) =< e,\delta(x) >$ for all $x \in X$ and then give the certain conditions for innerness of a d-derivation δ on (X,π_e). In the third section, we suppose that A is unital and e is an element of X such that ${}_X< e,e >= < e,e >_X = 1_A$ and define the map $*: X \to X$ by $x^* := e.< x,e >_X$ and we show that X is a C*-algebra with the product π_e and subalgebra of K(X).

Keywords: Hilbert C*-modules, Banach algebras, C*-algebras, d-derivation.

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