Shannon Wavelet Regularization Method for the Cauchy Problem Associated with the Helmholtz Equation

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Extended Abstract

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Introduction

Consider the Cauchy problem for the Helmholtz equation in an infinite "strip" domain as

$$\begin{cases} \Delta u(x,y)+\kappa^2 u(x,y)=0, & (x,y)\in(0,d]\times\mathbb{R},\\ u(0,y)=\varphi(y), & y\in\mathbb{R},\\ \partial_x u(0,y)=0, & y\in\mathbb{R}, \end{cases} \tag{1}$$

where $\Delta=\partial_{xx}+\partial_{yy}$ denotes the two-dimensional Laplace operator, $\kappa=\kappa_r+i\kappa_i\in\mathbb{C}$, indicating the number of wave, $i=\sqrt{-1}$, the imaginary unit and $d>0$. In the cases $\kappa_i=0$, $\kappa_r=0$ and $\kappa=0$, the problem (1) is called the Cauchy problem for the scalar Helmholtz equation, the Cauchy problem for the Yukawa equation and the Cauchy problem for the Laplace equation, respectively. The data $\varphi(\cdot)$ is measured based on physical observations, thus it can be inexactly computed by $\varphi_m(\cdot)$ satisfying

$$\|\varphi-\varphi_m\|_{L^2}\le\delta, \tag{2}$$

where δ represents the level of noise. The Cauchy problem for the Helmholtz equation (1) is the well-spring of many streams in both mathematical and technological problems due to the formal equivalence of the wave equation. The Helmholtz equation is a special kind of elliptic equations which has many practical applications in science and technology. For instance, this equation is able to describe the vibration of a structure, the acoustic cavity problem, the radiation waves, linearization of the Poisson-Boltzmann equation, and heat conduction in a fin. The inverse problem given by Eq. (1) is well-known to be extremely ill-posed in the sense of Hadamard. Therefore, it is desirable to design an effective strategy to retrieve the solution of problem (1). In this paper, a regularization strategy based on Shannon wavelet methods is proposed to solve the proposed inverse problem. The Sahnnon wavelet is compact and well-local in the frequency space, moreover, it is able to prevent the solution in the presence of high frequency perturbations. By applying the Shannon wavelet projection to the considered problem, an a-proper regularization parameter is formulated to approximate solution of the problem (1).

Material and methods

Let ϕ be the Shannon scaling function in one-dimension and is expressed by its Fourier transformation via

$$\hat{\phi}(\omega) = \begin{cases} \frac{1}{\sqrt{2\pi}}, & |\omega| \le \pi, \\ 0, & \text{otherwise}, \end{cases}$$

while the corresponding wavelet function ψ -called "mother wavelet"- is given by

$$\hat{\psi}(\omega) = \begin{cases} \frac{1}{\sqrt{2\pi}} e^{-i\frac{\omega}{2}}, & \pi \le |\omega| \le 2\pi, \\ 0, & \text{otherwise}. \end{cases}$$

We see that for all $j, k \in \mathbb{Z}$, the functions $\psi_{J,k}(x) := 2^{J/2}\psi(2^J x - k)$ constitute an orthonormal basis of the Hilbert space $L^2(\mathbb{R})$. For any $f(\cdot) \in L^2(\mathbb{R})$, the orthogonal projection on elements of Shannon multiresolution analysis is defined as

$$P_J f(x) = \sum_{k \in \mathbb{Z}} \langle f, \phi_{J,k} \rangle \phi_{J,k}(x).$$

The central aim of the proposed method is to retrieve the approximate solution-called regularized solution- from the true solution under imposing an a-priori information as

$$\|u(d,\cdot)\|_{H^p} \le M, \tag{3}$$

where $M > 0$ is a non-dimensional a -priori bound. In this paper, we introduce the Shannon wavelet solution for the problem (1) as

$$u_J(x,y) = \sum_{k \in \mathbb{Z}} c_k(x)\phi_{J,k}(y).$$

Results and discussion

Central Theorem: Suppose that the assumptions (2) and (3) are hold. Then the regularized solution $u_J^\delta(x,\cdot)$ is stable in the sense of Hadamard for the regularization parameter

$$J = J(\delta, M) = \Big[\frac{1}{2}\log_2\Big(\frac{1}{\pi^2}\Big(\ln\Big(\big(\frac{M}{\delta}\big)^{1/d}\big(\frac{1}{d}\ln\frac{M}{\delta}\big)^{-p/d}\Big)\Big)^2 + \big(\frac{\kappa}{\pi}\big)^2\Big)\Big].$$

Furthermore, the following inequality is satisfied

$$\| u(x,\cdot) - u_J^\delta(x,\cdot)\|_{L^2} \le 2M^{\frac{x}{d}}\delta^{1-\frac{x}{d}}\Big(\frac{1}{d}\ln\frac{M}{\delta}\Big)^{-p\frac{x}{d}}(1+o(1)), \quad \text{as} \quad \delta \to 0.$$

This strategy produces an optimal stable estimate of the so-called Holder-Logarithmic type under an a-priori condition and also predicts that when $p > 0$, the convergence rate of Logarithmic type at $x = d$ is faster than one in the Holder type. Thus the convergence rate of the proposed regularization method for the Cauchy problem for the Helmholtz equation is of order optimal and therefore there is no computational method to retrieve the solution of the considered a problem that would satisfy the better L^2-scale. Consequently, the wavelet methods must be applicable and new powerful technique to treat ill-posed problems.

Finally, we test some prototype smooth and non-smooth examples by using the proposed scheme which in turn corroborate our theoretical analysis. From numerical examples, one is able to judge the proposed method has a good agreement with the regularized and exact solution.

Conclusion

The following conclusions were drawn from this research.

- From the standpoint of theoretical analysis, we rigorously proved the well-posedness of the problem (1) in the sense of Hadamard in an appropriate normed space.
- The regularization parameter can be derived via an optimization process, sophisticatedly.
- Applying the Shannon wavelet regularization strategy, a regularized solution for the problem (1) and a Holder-Logarithmic type explicit error estimate of order optimal based on a-priori parameter are derived.
- The provided numerical simulations illustrate that the proposed strategy is very effective and successful to solve the proposed inverse problem (1).

Keywords: Cauchy problem; Helmholtz equation; Shannon wavelet; Regularization.

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