Approximation in Banach Spaces of Vector-Valued Lipschitz Functions

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Extended Abstract

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Introduction

Let (X, d) be a nonempty metric space, $(S, \|\cdot\|_S)$ be a complex Banach space and $0< \alpha \leq 1$. By $Lip_\alpha(X, S)$ we denote the space of all S-valued bounded functions f on X for which

is finite and by $lip_\alpha(X, S)$ we denote the subspace of $Lip_\alpha(X, S)$ consisting of functions f for which

For $f\in Lip_\alpha(X, S)$ define $\|f\|_{\alpha,S}= \|f\|_X+p_{\alpha,S}(f)$ where $\|\cdot\|_X$ is the supremum norm on X. Then $(Lip_\alpha(X, S), \|\cdot\|_{\alpha,S})$ is a Banach space and $lip_\alpha(X, S)$ is a closed subspace of $Lip_\alpha(X, S)$. When $S = \mathbb{C}$, we write simply $Lip_\alpha(X)$ and $lip_\alpha(X)$ for $Lip_\alpha(X, \mathbb{C})$ and $lip_\alpha(X, \mathbb{C})$, respectively. In this paper, by considering the space of measures on X whose values lie in the dual space S^* of S, we adopt a similar method as in [1] for vector-valued Lipschitz functions to give a criterion for the density of a subspace of $lip_\alpha(X, S)$. Then we conclude that $Lip_1([a,b], S)$ is dense in $lip_\alpha([a,b], S)$. Finally using Bochner spaces, S^*-valued measure theory and duality, we show that $C^1([a,b], S)$ is dense in $lip_\alpha([a,b], S)$.

Main Results

Let X be a locally compact Hausdorff space, and S a complex Banach space. The space of all bounded continuous S-valued functions on X will be denoted by C(X,S) and the subspace of C(X,S) consisting of all functions vanishing at infinity by $C_0(X,S)$. These spaces, equipped by supremum norm $\|f\|_X=\sup_{x\in X} \|f(x)\|_S$ for $f\in C(X,S)$, are Banach spaces. In the case $S = \mathbb{C}$, we will denote these spaces by C(X) and $C_0(X)$.

The space of all complex regular Borel measures on X will be denoted by M(X), and the dual space of S by S^*. It will be denoted by $M(X, S^*)$, the space of all S^*-valued Borel measures μ on X satisfying the following conditions:

i. $\langle s,\mu(\cdot)\rangle \in M(X)$ for every $s \in S$;

ii. There exists a constant C such that $\sup|\sum\langle s_i, \mu(A_i)\rangle|\leq C$ where the supremum is taken over all partitions of X into a finite number of disjoin Borel sets $\{A_i\}$ and all finite collections of elements $\{s_i\}$ in S such that $\|s_i\|_S\leq 1$.

For $\in \mu M(X, S^*)$ and $f\in C(X,S)$, the integral $\int_X f\, d\mu$ is of the Riemann-Stieltjes type obtained by taking the limit under successive refinements of sums of the form $\sum\langle f(x_i) , \mu(A_i)\rangle$, where $\{A_i\}$ is a finite partition of X into disjoint Borel sets and $\{x_i\}$ is a sequence in X such that $x_i\in A_i$.

Let (X, d) be a compact metric space, and set $V=\{(x,y)\in X\times X : x\neq y\}$ and $\tilde{X}=X\cup V$, so that $\tilde{X}$ is a locally compact space. Fixing $0<\alpha\leq 1$, for $f\in C(X,S)$ define $\tilde{f}$ on $\tilde{X}$ by

Then $\tilde{f}\in C(\tilde{X},S)$ and the mapping $f\mapsto \tilde{f}$ is a linear isometry from $(Lip_\alpha(X, S), \|\cdot\|_{\alpha,S})$ into $(C(\tilde{X},S), □.□_{\tilde{X}})$, where $□.□_{\tilde{X}}=\|\cdot\|_X+\|\cdot\|_V$. It can be easily seen that the image of $lip_\alpha(X, S)$ is contained in $C_0(\tilde{X},S)$.

Lemma 1. For every $\mu \in M(\tilde{X}, S^*)$ and $\varepsilon > 0$, there exists $\nu \in M(X, S^*)$ such that

$$\left|\int_{\tilde{X}} \tilde{f}\, d\mu - \int_X f\, d\nu\right| < \varepsilon\|f\|_{\alpha,S}, \qquad f \in Lip_\alpha(X, S) .$$

Lemma 2. Let $\nu \in M(X, S^*)$ and $\varepsilon > 0$ .Then there exists a measure $\lambda \in M(X, S^*)$ with finite support such that

$$\left|\int_X f\, d\nu - \int_X f\, d\lambda\right| < \varepsilon \|f\|_{\alpha,S}, \qquad f \in lip_\alpha(X, S).$$

Applying two above lemmas, we obtain the following approximation theorem.

Theorem 3. Let L be a linear subspace of $lip_\alpha(X, S)$. Suppose that there is a constant C such that for every finite subset E of X and each $f \in lip_\alpha(X, S)$, there exists $g\in L$ such that $f= g$ on E and $\|g\|_{\alpha,S} \leq C\|f\|_{\alpha,S}$. Then L is dense in $lip_\alpha(X, S)$.

The following theorem is similar to [1, Lemma 3.3]. The proof given in [1] for the scalar case does not apply here though.

Theorem 4. For every $f \in lip_\alpha([a,b] , S)$ and for every finite subset E of [a,b], there exists a function $g \in Lip_1([a,b] , S)$ such that f=g on E and $\|g\|_{\alpha,S}\leq 3\|f\|_{\alpha,S}$.

As an immediate consequence of Theorems 3 and ٤ we obtain:

Corollary 5. $Lip_1([a,b] , S)$ is dense in $lip_\alpha([a,b] , S)$.

Corollary 6. Let $0<\alpha<\beta<1$. Then $Lip_\beta([a,b] , S)$ and $lip_\beta([a,b] , S)$ are dense in $lip_\alpha([a,b] , S)$.

Next, we show that $C^1([a,b],S)$ is dense in $lip_\alpha([a,b] , S)$. It is easy to see that $C^1([a,b],S) \subseteq lip_\alpha([a,b] , S)$ and

where I = [a,b]. Thus, $C^1([a,b],S)$ is a closed subspace of $Lip_1([a,b] , S)$. In fact, $C^1([a,b], S)$ is a proper subspace of $Lip_1([a,b] , S)$. Using Lemmas 1 and٢ we obtain the following approximation theorem.

Theorem 7. Let $0<\alpha<1$. Then $C^1([a,b],S)$ is dense in $lip_\alpha([a,b] , S)$.

Keywords: Vector-valued Lipschitz space; Continuously differentiable vector-valued function; Vector-valued measure; Approximation.

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